Cubic spline wavelets with complementary boundary conditions

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Abstract

We propose a new construction of a stable cubic spline-wavelet basis on the interval satisfying complementary boundary conditions of the second order. It means that the primal wavelet basis is adapted to homogeneous Dirichlet boundary conditions of the second order, while the dual wavelet basis preserves the full degree of polynomial exactness. We present quantitative properties of the constructed bases and we show superiority of our construction in comparison to some other known spline wavelet bases in an adaptive wavelet method for the partial differential equation with the biharmonic operator.

Keywords: wavelet, cubic spline, complementary boundary conditions, homogeneous Dirichlet boundary conditions, condition number
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1. Introduction

In recent years wavelets have been successfully used for solving partial differential equations [2, 11, 12, 16, 27] as well as integral equations [22, 24, 25], both linear and nonlinear. Wavelet bases are useful in the numerical treatment of operator equations, because they are stable, enable high order-approximation, functions from Besov spaces have sparse representation in wavelet bases, condition numbers of stiffness matrices are uniformly bounded and matrices representing operators are typically sparse or quasi-sparse. The

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quantitative properties of wavelet methods strongly depend on the choice of a wavelet basis, in particular on its condition number. Therefore, a construction of a wavelet basis is always an important issue.

Wavelet bases on a bounded domain are usually constructed in the following way: Wavelets on the real line are adapted to the interval and then by tensor product technique to the n-dimensional cube. Finally by splitting the domain into overlapping or non-overlapping subdomains which are images of a unit cube under appropriate parametric mappings one can obtain a wavelet basis or a wavelet frame on a fairly general domain. Thus, the properties of the employed wavelet basis on the interval are crucial for the properties of the resulting bases or frames on a general domain.

In this paper, we propose a construction of cubic spline wavelet basis on the interval that is adapted to homogeneous Dirichlet boundary conditions of the second order on the primal side and preserves the full degree of polynomial exactness on the dual side. Such boundary conditions are called complementary boundary conditions [18]. We compare properties of wavelet bases such as the condition number of the basis and the condition number of the corresponding stiffness matrix. Finally, quantitative behaviour of adaptive wavelet method for several boundary-adapted cubic spline wavelet bases is studied.

First of all, we summarize the desired properties of a constructed basis:

- *Polymial exactness.* Since the primal basis functions are cubic B-splines, the primal multiresolution analysis has polynomial exactness of order four. The dual multiresolution analysis has polynomial exactness of order six. As a consequence, the primal wavelets have six vanishing moments.
- *Riesz basis property.* The functions form a Riesz basis of the space $L^2([0,1])$ and if scaled properly they form a Riesz basis of the space $H_0^2([0,1])$.
- *Locality.* The primal and dual basis functions are local, see definition of locality below. Then the corresponding decomposition and reconstruction algorithms are simple and fast.
- *Biorthogonality.* The primal and dual wavelet bases form a biorthogonal pair.
- *Smoothness.* The smoothness of primal and dual wavelet bases is another desired property. It ensures the validity of norm equivalences.
- *Closed form.* The primal scaling functions and wavelets are known in

the closed form. It is a desirable property for the fast computation of integrals involving primal scaling functions and wavelets.

- *Complementary boundary conditions.* Our wavelet basis satisfy complementary boundary conditions of the second order.
- *Well-conditioned bases.* Our objective is to construct a well conditioned wavelet basis.

Many constructions of cubic spline wavelet or multiwavelet bases on the interval have been proposed in recent years. In [5, 17, 26] cubic spline wavelets on the interval were constructed. In [14] cubic spline multiwavelet bases were designed and they were adapted to complementary boundary conditions of the second order in [28]. In this case dual functions are known and are local. Cubic spline wavelet bases were also constructed in [1, 9, 20, 21]. A construction of cubic spline multiwavelet basis was proposed in [19] and this basis was already used for solving differential equations in [8, 23]. However, in these cases duals are not known or are not local. Locality of duals are important in some methods and theory, let us mention construction of wavelet bases on general domain [18], adaptive wavelet methods especially for nonlinear equations, data analysis, signal and image processing. A general method of adaptation of biorthogonal wavelet bases to complementary boundary conditions was presented in [18], but this method often leads to very badly conditioned bases.

This paper is organized as follows: In Section 2 we briefly review the concept of wavelet bases. In Section 3 we propose a construction of primal and dual scaling bases. The refinement matrices are computed in Section 4 and in Section 5 primal and dual wavelets are constructed. Quantitative properties of constructed bases and other known cubic spline wavelet and multiwavelet bases are studied in Section 6. In Section 7 we compare the number of basis functions and the number of iterations needed to resolve the problem with desired accuracy for our bases and bases from [28]. A numerical example is presented for an equation with the biharmonic operator in two dimensions.

2. Wavelet bases

This section provides a short introduction to the concept of wavelet bases in Sobolev spaces. We consider the domain $\Omega \subset \mathbb{R}^d$ and the Sobolev space or its subspace $H \subset H^s(\Omega)$ for nonnegative integer s with an inner product

$\langle \cdot, \cdot \rangle_H$, a norm $\|\cdot\|_H$ and a seminorm $|\cdot|_H$. In case $s = 0$ we consider the space $L^2(\Omega)$ and we denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the L^2-inner product and the L^2-norm, respectively. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j, k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale* or a *level*. Let

$$l^2(\mathcal{J}) := \left\{ \mathbf{v} : \mathcal{J} \to \mathbb{R}, \sum_{\lambda \in \mathcal{J}} |\mathbf{v}_\lambda|^2 < \infty \right\}. \tag{1}$$

A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a *wavelet basis* of H, if

i) Ψ is a *Riesz basis* for H, i.e. the closure of the span of Ψ is H and there exist constants $c, C \in (0, \infty)$ such that

$$c \|\mathrm{b}\|_{l^2(\mathcal{J})} \le \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \le C \|\mathrm{b}\|_{l^2(\mathcal{J})}, \quad \mathrm{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}). \tag{2}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies (2)}\}$, $C_\psi := \inf\{C : C \text{ satisfies (2)}\}$ are called *Riesz bounds* and *cond* $\Psi = C_\psi / c_\psi$ is called the *condition number* of Ψ.

ii) The functions are *local* in the sense that $\operatorname{diam}(\Omega_\lambda) \le \mathrm{C}2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where Ω_λ is the support of ψ_λ, and at a given level j the supports of only finitely many wavelets overlap at any point $x \in \Omega$.

By the Riesz representation theorem, there exists a unique family $\tilde{\Psi} = \left\{ \tilde{\psi}_\lambda, \lambda \in \tilde{\mathcal{J}} \right\} \subset H$ biorthogonal to Ψ, i.e.

$$\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle_H = \delta_{i,j} \delta_{k,l}, \quad \text{for all} \quad (i,k) \in \mathcal{J}, \quad (j,l) \in \tilde{\mathcal{J}}. \tag{3}$$

This family is also a Riesz basis for H. The basis Ψ is called a *primal* wavelet basis, while $\tilde{\Psi}$ is called a *dual* wavelet basis.

In many cases, the wavelet system Ψ is constructed with the aid of a multiresolution analysis. A sequence $\mathcal{V} = \{V_j\}_{j \ge j_0}$, of closed linear subspaces $V_j \subset H$ is called a *multiresolution* or *multiscale analysis*, if

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots H \tag{4}$$

and $\cup_{j \ge j_0} V_j$ is complete in H.

The nestedness and the closedness of the multiresolution analysis implies the existence of the *complement spaces* W_j such that $V_{j+1} = V_j \oplus W_j$.

We now assume that V_j and W_j are spanned by sets of basis functions

$$\Phi_j := \{\phi_{j,k}, k \in \mathcal{I}_j\}, \quad \Psi_j := \{\psi_{j,k}, k \in \mathcal{J}_j\}, \tag{5}$$

where $\mathcal{I}_j$ and $\mathcal{J}_j$ are finite or at most countable index sets. We refer to $\phi_{j,k}$ as *scaling functions* and $\psi_{j,k}$ as *wavelets*. The multiscale basis is given by $\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j$ and the wavelet basis of H is obtained by $\Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j$. The dual wavelet system $\tilde{\Psi}$ generates a dual multiresolution analysis $\tilde{\mathcal{V}}$ with a dual scaling basis $\tilde{\Phi}_{j_0}$.

Polynomial exactness of order $N \in \mathbb{N}$ for the primal scaling basis and of order $\tilde{N} \in \mathbb{N}$ for the dual scaling basis is another desired property of wavelet bases. It means that

$$\mathbb{P}_{N-1}(\Omega) \subset V_j, \quad \mathbb{P}_{\tilde{N}-1}(\Omega) \subset \tilde{V}_j, \quad j \geq j_0, \tag{6}$$

where $\mathbb{P}_m(\Omega)$ is the space of all algebraic polynomials on Ω of degree less or equal to m.

By Taylor theorem, the polynomial exactness of order $\tilde{N}$ on the dual side is equivalent to $\tilde{N}$ vanishing wavelet moments on the primal side, i.e.

$$\int_\Omega P(x)\, \psi_\lambda(x)\, dx = 0, \quad P \in \mathbb{P}_{\tilde{N}-1}, \ \psi_\lambda \in \bigcup_{j \geq j_0} \Psi_j. \tag{7}$$

3. Construction of Scaling Functions

We propose a new cubic spline wavelet basis with six vanishing wavelet moments satisfying homogeneous Dirichlet boundary conditions of order two. Six vanishing wavelet moments on the primal side is equivalent to the polynomial exactness of order six on the dual side. We choose polynomial exactness of this order, because the dual scaling function of order four does not belong to $L^2(\mathbb{R})$ and the polynomial exactness of order greater than six leads to a larger support of primal wavelets which makes the computation more expensive.

The first step is the construction of primal scaling functions on the unit interval. Primal scaling basis is formed by cubic B-splines on the knots t_k^j defined by

$$t_{-2}^j = t_{-1}^j := 0, \quad t_0^j := \frac{1}{2^{j+1}}, \quad t_k^j := \frac{k}{2^j}, \quad k = 1, \ldots 2^j - 1, \tag{8}$$

$$t_{2^j}^j := \frac{2^{j+1}-1}{2^{j+1}}, \quad t_{2^j+1}^j = t_{2^j+2}^j := 1.$$

The corresponding cubic B-splines are defined by

$$B_k^j(x) := \left(t_{k+4}^j - t_k^j\right) \left[t_k^j, \ldots, t_{k+4}^j\right]_t (t-x)_+^3, \quad x \in [0,1],$$

where $(x)_+ := \max\{0, x\}$ and $[t_1, \ldots t_N]_t f$ is the N-th divided difference of f. The set $\Phi_j := \{\phi_{j,k}, k = -2, \ldots, 2^j - 2\}$ of primal scaling functions is simply given by

$$\phi_{j,k} := 2^{j/2} B_k^j, \quad k = -2, \ldots, 2^j - 2, \quad j \geq 0. \tag{9}$$

Thus there are $2^j - 5$ inner scaling functions and 3 boundary functions at each edge. The inner functions are translations and dilations of a function ϕ which corresponds to the primal scaling function constructed by Cohen, Daubechies, and Feauveau in [10]. Note that the primal scaling basis differs from the primal scaling basis constructed in [4, 5, 17, 26], because there are additional knots $\frac{1}{2^{j+1}}$ and $\frac{2^{j+1}-1}{2^{j+1}}$.

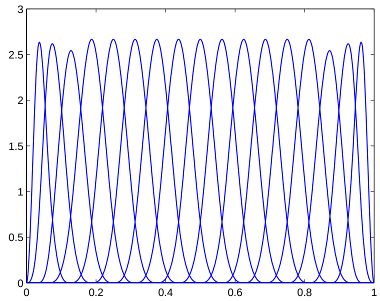

Figure 1: Primal scaling functions for the scale $j = 4$.

The desired property of a dual scaling basis $\tilde{\Phi}$ is the biorthogonality to Φ and the polynomial exactness of order six. Let $\tilde{\phi}$ be the dual scaling function which was designed by Cohen, Daubechies, and Feauveau in [10] and which is shifted so that $\tilde{\phi}$ is orthogonal to ϕ, i.e. its support is $[-5, 9]$. It is known that there exist sequences $\{h_k\}_{k=0}^4$ and $\{\tilde{h}_k\}_{k=-5}^9$ such that the functions ϕ and $\tilde{\phi}$ satisfy the *refinement equations*

$$\phi(x) = \sum_{k=0}^{4} h_k \phi(2x-k), \quad \tilde{\phi}(x) = \sum_{k=-5}^{9} \tilde{h}_k \tilde{\phi}(2x-k), \quad x \in \mathbb{R}. \tag{10}$$

The parameters h_k and $\tilde{h}_k$ are called *scaling coefficients*.

In the sequel, we assume that $j \geq j_0 := 4$. We define inner scaling functions as translations and dilations of $\tilde{\phi}$:

$$\theta_{j,k} = 2^{j/2} \tilde{\phi} \left(2^j \cdot -k \right), \quad k = 5, \ldots, 2^j - 9. \tag{11}$$

There will be two types of basis functions at each boundary. In the following, it will be convenient to abbreviate the boundary and inner index sets by

$$\begin{aligned} \mathcal{I}_j^{L,1} &= \{-2, \ldots, 3\}, \quad \mathcal{I}_j^{L,2} = \{4\}, \quad \mathcal{I}_j^0 = \left\{5, \ldots, 2^j - 9\right\}, \\ \mathcal{I}_j^{R,2} &= \left\{2^j - 8\right\}, \quad \mathcal{I}_j^{R,1} = \left\{2^j - 7, \ldots, 2^j - 2\right\}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \mathcal{I}_j^{L} &= \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2} = \{-2, \ldots, 4\}, \\ \mathcal{I}_j^{R} &= \mathcal{I}_j^{R,2} \cup \mathcal{I}_j^{R,1} = \left\{2^j - 8, \ldots, 2^j - 2\right\}, \\ \mathcal{I}_j &= \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2} \cup \mathcal{I}_j^0 \cup \mathcal{I}_j^{R,2} \cup \mathcal{I}_j^{R,1} = \left\{-2, \ldots, 2^j - 2\right\}. \end{aligned} \tag{13}$$

Basis functions of the first type are defined to preserve polynomial exactness and the nestedness of multiresolution spaces by the same way as in [17]:

$$\theta_{j,k}(x) = 2^{j/2} \sum_{l=-8}^{4} \langle p_{k+2}, \phi(\cdot - l) \rangle \tilde{\phi} \left(2^j x - l \right), \quad k \in \mathcal{I}_j^{L,1}, \quad x \in [0,1], \tag{14}$$

where $\{p_0, \ldots, p_5\}$ is a monomial basis of $\mathbb{P}_5([0,1])$, i.e. $p_i(x) = x^i$, $x \in [0,1]$, $i = 0, \ldots, 5$.

The definition of basis functions of the second type is a delicate task, because the low condition number and the nestedness of the multiresolution spaces have to be preserved. This means that the relation $\theta_{j,4} \in \tilde{V}_j \subset \tilde{V}_{j+1}$ should hold. Therefore we define $\theta_{j,4}$ as linear combinations of functions that are already in $\tilde{V}_{j+1}$. To obtain well-conditioned basis, we define a function $\theta_{j,4}$ which is close to $\tilde{\phi}_{j,4}^{\mathbb{R}} := 2^{j/2} \tilde{\phi}(2^j \cdot -4)$, because $\tilde{\phi}_{j,4}^{\mathbb{R}}$ is biorthogonal to the inner primal scaling functions and the condition of $\left\{ \tilde{\phi}_{j,4}^{\mathbb{R}}, k \in \mathcal{I}_j^{L,2} \cup \mathcal{I}_j^0 \right\}$ is close to the condition of the set of inner dual basis functions.

For this reason, we define the basis function of the second type by

$$\theta_{j,4}(x) = 2^{j/2} \sum_{l=-3}^{9} \tilde{h}_l \tilde{\phi}\left(2^{j+1}x - 8 - l\right), \quad x \in [0,1], \tag{15}$$

where $\tilde{h}_i$ are the scaling coefficients corresponding to the scaling function $\tilde{\phi}$. Then $\theta_{j,4}$ is close to $\tilde{\phi}^{\mathbb{R}}_{j,4}$ restricted to the interval $[0,1]$, because by (10) we have

$$\tilde{\phi}^{\mathbb{R}}_{j,4}(x) = 2^{j/2} \sum_{l=-5}^{9} \tilde{h}_l \tilde{\phi}\left(2^{j+1}x - 8 - l\right), \quad x \in [0,1]. \tag{16}$$

Figure 2 shows the functions $\theta_{4,4}$ and $\tilde{\phi}^{\mathbb{R}}_{4,4}$.

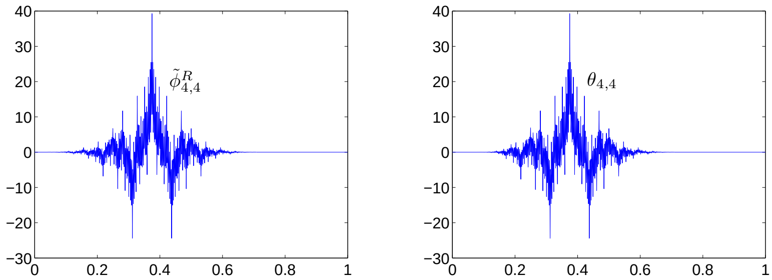

Figure 2: The functions $\tilde{\phi}^{\mathbb{R}}_{4,4}$ and $\theta_{4,4}$.

The boundary functions at the right boundary are defined to be symmetric with the left boundary functions:

$$\theta_{j,k}(x) = \theta_{j,2^j-4-k}(1-x), \quad x \in [0,1], \quad k \in \mathcal{I}_j^R. \tag{17}$$

It is easy to see that

$$\theta_{j+1,k}(x) = \sqrt{2}\,\theta_{j,k}(2x), \quad x \in [0,1], \quad k \in \mathcal{I}_j^L, \tag{18}$$

for left boundary functions and

$$\theta_{j+1,k}(1-x) = \sqrt{2}\,\theta_{j,k}(1-2x), \quad x \in [0,1], \quad k \in \mathcal{I}_j^R, \tag{19}$$

for right boundary functions.

Since the set $\Theta_j := \{\theta_{j,k}, k \in \mathcal{I}_j\}$ is not biorthogonal to Φ_j, we derive a new set

$$\tilde{\Phi}_j := \left\{\tilde{\phi}_{j,k}, k \in \mathcal{I}_j\right\} \tag{20}$$

from Θ_j by biorthogonalization. Let

$$\mathbf{Q}_j = \left(\langle \phi_{j,k}, \theta_{j,l} \rangle\right)_{k,l \in \mathcal{I}_j}. \tag{21}$$

We verify numerically that $\mathbf{Q}_j$ is invertible. Viewing $\tilde{\Phi}_j$ and Θ_j as column vectors we define

$$\tilde{\Phi}_j := \mathbf{Q}_j^{-T} \Theta_j. \tag{22}$$

Then $\tilde{\Phi}_j$ is biorthogonal to Φ_j, because

$$\left\langle \Phi_j, \tilde{\Phi}_j \right\rangle = \left\langle \Phi_j, \mathbf{Q}_j^{-T} \Theta_j \right\rangle = \mathbf{Q}_j \mathbf{Q}_j^{-1} = \mathbf{I}_{\# \mathcal{I}_j}, \tag{23}$$

where the symbol $\#$ denotes the cardinality of the set and $\mathbf{I}_m$ denotes the identity matrix of the size $m \times m$.

Remark 1. General approach of adapting wavelet bases to the unit interval was proposed in [18]. The idea is to remove certain boundary scaling functions to achieve homogeneous boundary conditions on the primal side. Then it is necessary to have the same number of basis functions on the dual side. Therefore an appropriate number of inner dual functions is used for the definition of boundary dual generators in formula (14). Applying this approach to cubic spline basis constructed in [5] and basis constructed in [26] we obtain the same resulting basis, because these constructions differs in the definition of some functions which are discarded during adaptation to complementary boundary conditions of the second order. Unfortunately, this basis has large condition number, although the starting basis in [5] is well conditioned. Its quantitative properties are presented in Section 6.

4. Refinement matrices

From the nestedness and the closedness of multiresolution spaces it follows that there exist *refinement matrices* $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ such that

$$\Phi_j = \mathbf{M}_{j,0}^T \Phi_{j+1}, \quad \tilde{\Phi}_j = \mathbf{M}_{j,1}^T \tilde{\Phi}_{j+1}. \tag{24}$$

Due to the length of support of primal scaling functions, the refinement matrix $\mathbf{M}_{j,0}$ has the following structure:

$$\mathbf{M}_{j,0} = \left(\begin{array}{c|c|c} \mathbf{M}_L & & \\ \cline{1-1} & \mathbf{A}_j & \\ \cline{3-3} & & \mathbf{M}_R \end{array} \right). \tag{25}$$

where $\mathbf{A}_j$ is a $(2^{j+1} - 5) \times (2^j - 5)$ matrix given by

$$\begin{aligned} (\mathbf{A}_j)_{m,n} &= \frac{h_{m+1-2n}}{\sqrt{2}}, \quad n = 1, \ldots, 2^j - 5,\ 0 \le m + 1 - 2n \le 4, \\ &= 0, \qquad\qquad\qquad \text{otherwise,} \end{aligned} \tag{26}$$

where h_m are primal scaling coefficients (10), and $\mathbf{M}_L$, $\mathbf{M}_R$ are given by

$$\mathbf{M}_L = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ \frac{7}{8} & \frac{1}{8} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{5} & \frac{2}{5} \\ 0 & \frac{3}{20} & \frac{29}{40} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{8} \end{pmatrix}, \quad \mathbf{M}_R = \mathbf{M}_L^{\updownarrow}. \tag{27}$$

The symbol $\mathbf{M}^{\updownarrow}$ denotes a matrix that results from a matrix $\mathbf{M}$ by reversing the ordering of rows and columns. To compute the refinement matrix corresponding to the dual scaling functions, we need to identify first the structure of refinement matrices $\mathbf{M}_{j,0}^{\Theta}$ corresponding to Θ:

$$\mathbf{M}_{j,0}^{\Theta} = \left(\begin{array}{c|c|c} \mathbf{M}_L^{\Theta} & & \\ \cline{1-1} & \tilde{\mathbf{A}}_j & \\ \cline{3-3} & & \mathbf{M}_R^{\Theta} \end{array} \right), \tag{28}$$

where $\mathbf{M}_L^\Theta$ and $\mathbf{M}_R^\Theta$ are blocks 21×7 and $\tilde{\mathbf{A}}_j$ is a matrix of the size $(2^{j+1}-13)\times (2^j-13)$ given by

$$\begin{aligned}\left(\tilde{\mathbf{A}}_j\right)_{m,n} &= \frac{\tilde{h}_{m-2n-4}}{\sqrt{2}}, \quad n=1,\dots 2^j-13,\ -1\le m-2n\le 13, \\ &= 0, \qquad\qquad\qquad \text{otherwise,}\end{aligned} \tag{29}$$

where $\tilde{h}_m$ are dual scaling coefficients (10). The refinement coefficients for the left boundary functions of the first type are computed according to the proof of Lemma 3.1 in [17]. The refinement coefficients for the left boundary functions of the second type are given by definition (15). The matrix $\mathbf{M}_R^\Theta$ can be computed by the similar way. Since

$$\tilde{\Phi}_j = \mathbf{Q}_j^{-T}\Theta_j = \mathbf{Q}_j^{-T}\left(\mathbf{M}_{j,0}^\Theta\right)^T\Theta_{j+1} = \mathbf{Q}_j^{-T}\left(\mathbf{M}_{j,0}^\Theta\right)^T\mathbf{Q}_{j+1}^T\tilde{\Phi}_{j+1}, \tag{30}$$

the refinement matrix $\tilde{\mathbf{M}}_{j,0}$ corresponding to the dual scaling basis $\tilde{\Phi}_j$ is given by

$$\tilde{\mathbf{M}}_{j,0} = \mathbf{Q}_{j+1}\mathbf{M}_{j,0}^\Theta\mathbf{Q}_j^{-1}. \tag{31}$$

5. Construction of wavelets

Our next goal is to determine the corresponding single-scale wavelet bases Ψ_j. It is directly connected to the task of determining an appropriate matrices $\mathbf{M}_{j,1}$ such that

$$\Psi_j = \mathbf{M}_{j,1}^T\Phi_{j+1}. \tag{32}$$

We follow a general principle called *stable completion* which was proposed in [3]. This approach was already used in [5, 17, 26]. In our case, however, the initial stable completion can not be found by the same way, because it leads to singular matrices.

Definition 1. Any $\mathbf{M}_{j,1}: l^2(\mathcal{J}_j)\to l^2(\mathcal{I}_{j+1})$ is called a *stable completion* of $\mathbf{M}_{j,0}$, if

$$\|\mathbf{M}_j\|_{l^2(\mathcal{I}_{j+1})\to l^2(\mathcal{I}_{j+1})} = O(1)\,,\ \left\|\mathbf{M}_j^{-1}\right\|_{l^2(\mathcal{I}_{j+1})\to l^2(\mathcal{I}_{j+1})} = O(1)\,,\quad j\to\infty, \tag{33}$$

where $\mathbf{M}_j := (\mathbf{M}_{j,0},\mathbf{M}_{j,1})$.

The idea is to determine first an initial stable completion and then to project it to the desired complement space W_j. This is summarized in the following theorem [3].

Theorem 2. *Let Φ_j and $\tilde{\Phi}_j$ be a primal and a dual scaling basis, respectively. Let $\mathbf{M}_{j,0}$ and $\tilde{\mathbf{M}}_{j,0}$ be refinement matrices corresponding to these bases. Suppose that $\check{\mathbf{M}}_{j,1}$ is some stable completion of $\mathbf{M}_{j,0}$ and $\check{\mathbf{G}}_j = \check{\mathbf{M}}_j^{-1}$. Then*

$$\mathbf{M}_{j,1} := \left(\mathbf{I} - \mathbf{M}_{j,0}\tilde{\mathbf{M}}_{j,0}^T\right)\check{\mathbf{M}}_{j,1} \tag{34}$$

is also a stable completion and $\mathbf{G}_j = \mathbf{M}_j^{-1}$ has the form

$$\mathbf{G}_j = \begin{pmatrix} \tilde{\mathbf{M}}_{j,0}^T \\ \check{\mathbf{G}}_{j,1} \end{pmatrix}. \tag{35}$$

Moreover, the collections

$$\Psi_j := \mathbf{M}_{j,1}^T \Phi_{j+1}, \quad \tilde{\Psi}_j := \check{\mathbf{G}}_{j,1}\tilde{\Phi}_{j+1}, \tag{36}$$

form biorthogonal systems

$$\left\langle \Psi_j, \tilde{\Psi}_j \right\rangle = \mathbf{I}, \quad \left\langle \Phi_j, \tilde{\Psi}_j \right\rangle = \left\langle \Psi_j, \tilde{\Phi}_j \right\rangle = \mathbf{0}. \tag{37}$$

To find the initial stable completion we use a factorization $\mathbf{M}_{j,0} = \mathbf{H}_j \mathbf{C}_j$, where

$$\mathbf{H}_j := \begin{pmatrix} \mathbf{H}_L & & \\ & \mathbf{H}_j^I & \\ & & \mathbf{H}_R \end{pmatrix}, \tag{38}$$

$$\mathbf{H}_L := \begin{pmatrix} 0.25 & 0 & 0 & 0 & 0 \\ 0.875 & 1 & 8 & 0 & 0 \\ 0.25 & 6 & 1 & 0 & 0 \\ 0 & 4.8 & 0 & 1 & 0 \\ 0 & 1.2 & 0 & 1.8125 & 2 \\ 0 & 0 & 0 & 1.25 & 1 \\ 0 & 0 & 0 & 0.3125 & 0 \end{pmatrix}, \quad \mathbf{H}_R := \mathbf{H}_L^{\updownarrow}, \tag{39}$$

Matrix $\left(\mathbf{H}_j^I\right)$ has the size $(2^{j+1}-7) \times (2^{j+1}-9)$. Its elements are given by:

$$\begin{aligned} \left(\mathbf{H}_j^I\right)_{mn} &:= 1, \quad 1 \le n \le 2^{j+1}-9,\ n \text{ odd}, m = n+1 \\ &:= h_{2,m-n+2}^I, \quad 1 \le n \le 2^{j+1}-9,\ n \text{ even}, -1 \le m-n \le 3, \\ &:= 0, \quad \text{otherwise}, \end{aligned} \tag{40}$$

where $h^I_{11} = h^I_{15} = 0.25$, $h^I_{12} = h^I_{14} = 1$, $h^I_{13} = 1.5$, and

$$\mathbf{C}_j := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{C}_L & & \\ & \mathbf{C}^I_j & \\ & & \mathbf{C}_R \end{pmatrix}, \quad \mathbf{C}_L := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} \end{pmatrix}, \tag{41}$$

$$\mathbf{C}_R := \mathbf{C}^{\updownarrow}_L, \quad \mathbf{C}^I_j := \begin{pmatrix} 0 & 0 & & 0 \\ 0 & 0 & & \\ b & 0 & & \\ 0 & 0 & & \\ 0 & b & & \\ \vdots & 0 & \ddots & \\ & & & b \\ & & & 0 \\ 0 & & & 0 \end{pmatrix}, \quad b := \frac{7}{8}. \tag{42}$$

The factorization corresponding to inner and boundary blocks is not the same as the factorization in [15]. Therefore by our approach we obtain new inner and boundary wavelets. We define

$$\mathbf{B}_j := \sqrt{2} \begin{pmatrix} \mathbf{B}_L & & \\ & \mathbf{B}^I_j & \\ & & \mathbf{B}_R \end{pmatrix}, \mathbf{B}_L := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{2} \end{pmatrix}, \mathbf{B}_R := \mathbf{B}^{\updownarrow}_L, \tag{43}$$

$$\mathbf{B}^I_j := \begin{pmatrix} 0 & 0 & b^{-1} & 0 & 0 & 0 & \dots & & & 0 \\ 0 & 0 & 0 & 0 & b^{-1} & 0 & \dots & & & 0 \\ & & & & & & \ddots & & & \\ & & & & & & & b^{-1} & 0 & 0 \end{pmatrix}, \tag{44}$$

and

$$\mathbf{F}_j := \begin{pmatrix} \mathbf{F}_L & & \\ & \mathbf{F}^I_j & \\ & & \mathbf{F}_R \end{pmatrix}, \tag{45}$$

$$\mathbf{F}_L := \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{F}_R := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{F}_j^I := \begin{pmatrix} 1 & 0 & & \\ 0 & 0 & & \\ 0 & 1 & & \\ \vdots & 0 & \ddots & \\ & & & 1 \end{pmatrix}. \tag{46}$$

The above findings can be summarized as follows.

Lemma 3. *The following relations hold:*

$$\mathbf{B}_j \mathbf{C}_j = \mathbf{I}_{\#\mathcal{I}_j}, \quad \mathbf{F}_j^T \mathbf{F}_j = \mathbf{I}_{2^j}, \quad \mathbf{B}_j \mathbf{F}_j = \mathbf{0}, \quad \mathbf{F}_j^T \mathbf{C}_j = \mathbf{0}. \tag{47}$$

Now we are able to define the initial stable completions of the refinement matrices $\mathbf{M}_{j,0}$.

Lemma 4. *Under the above assumptions, the matrices*

$$\check{\mathbf{M}}_{j,1} := \mathbf{H}_j \mathbf{F}_j, \quad j \geq j_0, \tag{48}$$

are uniformly stable completions of the matrices $\mathbf{M}_{j,0}$. *Moreover, the inverse*

$$\check{\mathbf{G}}_j = \begin{pmatrix} \check{\mathbf{G}}_{j,0} \\ \check{\mathbf{G}}_{j,1} \end{pmatrix} \tag{49}$$

of $\check{\mathbf{M}}_j = \left(\mathbf{M}_{j,0}, \check{\mathbf{M}}_{j,1} \right)$ *is given by* $\check{\mathbf{G}}_{j,0} = \mathbf{B}_j \mathbf{H}_j^{-1}, \quad \check{\mathbf{G}}_{j,1} = \mathbf{F}_j^T \mathbf{H}_j^{-1}$.

The proof of this lemma is straightforward and similar to the proof in [17]. Then using the initial stable completion $\check{\mathbf{M}}_{j,1}$ we are already able to contruct wavelets according to the Theorem 2. Left boundary wavelets are displayed at the Figure 5.

5.1. Decomposition of a scaling basis on a coarse scale

In the previous sections we assumed that the supports of the left and right boundary functions do not overlap and therefore the coarsest level was four. It might be too restrictive, especially in higher dimensions, because then there are many scaling functions. Here we decompose scaling basis Φ_4 into two parts Φ_3 and Ψ_3. It also improves the condition number of the basis. We construct wavelets on the level three to have four vanishing moments. Note that wavelets on other levels have six vanishing moments, but there the vanishing moments guaranties the smoothness of dual functions [10], and four

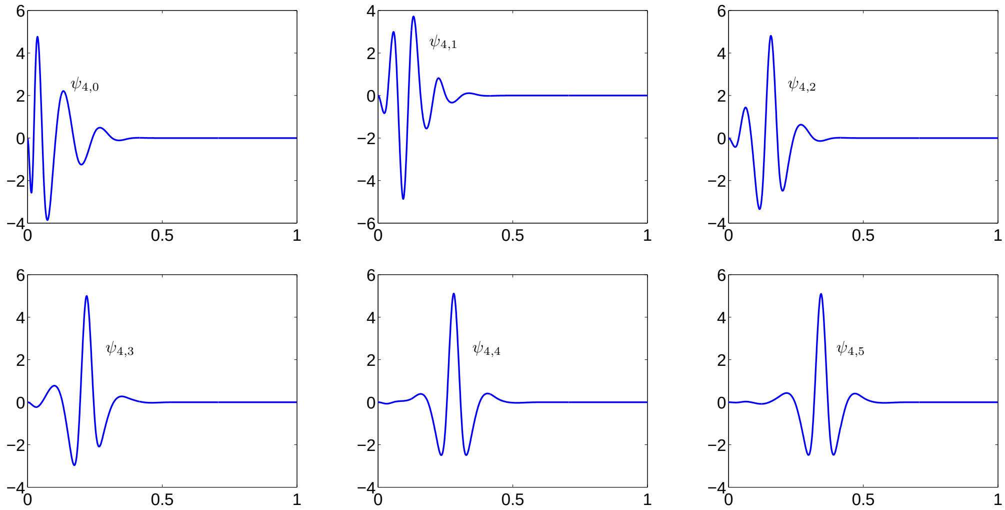

Figure 3: Left boundary wavelets for the scale $j = 4$.

vanishing moments for wavelets are sufficient in the most of the applications. Scaling functions in Φ_3 are defined by (9) for $j = 3$. Functions in Ψ_3 are defined by

$$\psi_{3,k}(x) := \frac{\left(B_{t_k}^8\right)^{(4)}(x)}{\left\|\left(B_{t_k}^8\right)^{(4)}\right\|}, \quad k = 1, \ldots, 8, \; x \in [0,1], \tag{50}$$

where $B_{t_k}^8$ is a B-spline of order eight on the sequence of knots t_k and $^{(4)}$ denotes the fourth derivative. The sequences of knots t_k are given by:

$$\begin{aligned}
t_1 &= [0, 0, 1/32, 1/16, 1/8, 2/8, 3/8, 4/8, 5/8]; \qquad (51)\\
t_2 &= [0, 1/32, 1/16, 1/8, 3/16, 2/8, 3/8, 4/8, 5/8];\\
t_3 &= [1/32, 1/16, 1/8, 2/8, 5/16, 3/8, 4/8, 5/8, 6/8];\\
t_4 &= [1/16, 1/8, 2/8, 3/8, 7/16, 4/8, 5/8, 6/8, 7/8];\\
t_5 &= [1/8, 2/8, 3/8, 4/8, 9/16, 5/8, 6/8, 7/8, 15/16];\\
t_6 &= [2/8, 3/8, 4/8, 5/8, 11/16, 6/8, 7/8, 15/16, 31/32];\\
t_7 &= [3/8, 4/8, 5/8, 6/8, 13/16, 7/8, 15/16, 31/32, 1];\\
t_8 &= [3/8, 4/8, 5/8, 6/8, 7/8, 15/16, 31/32, 1, 1];
\end{aligned}$$

Lemma 5. *Functions from the set* $\Phi_3 \cup \Psi_3$ *generate the same space as functions from the set* Φ_4, *i.e.* $\operatorname{span} \Phi_3 \cup \Psi_3 = \operatorname{span} \Phi_4$. *Functions* $\psi_{3,k}$, $k = 1, \ldots, 8$, *have four vanishing wavelet moments.*

Proof. Since Φ_4 is a basis of the space of all cubic splines on the knots $\mathbf{t}^4 = [0, 0, 1/32, 1/16, 2/16, \dots, 15/16, 31/32, 1, 1]$. Functions in Φ_3 are cubic splines on the subsets of these knots. Functions in Ψ_3 are also cubic splines, because they are fourth derivative of the spline of order eight, and they are defined on the subsets of knots $\mathbf{t}^4$. Therefore $\Phi_3 \cup \Psi_3 \subset \text{span}\, \Phi_4$.

Functions in Φ_3 are linearly independent. Function $\psi_{3,i}$ cannot be written as linear combination of functions from $\Phi_3 \cup \Psi_3 \setminus \{\psi_{3,i}\}$, because it is a cubic spline on sequence of the knots t_i containing an additional knot. Hence, $\Psi_3 \cup \Phi_3$ is a linearly independent subset of span Φ_4, which proves the first assertion.

To prove that the functions $\psi_{3,k}$, $k = 1, \dots, 8$, have four vanishing moments, we use the integration by parts. We obtain for $n = 0, \dots, 3$:

$$\int_0^1 x^n (B_{t_k}^8)^{(4)} (x)\, dx = \left[x^n \left(B_{t_k}^8 \right)^{(3)} (x) \right]_0^1 - \int_0^1 n x^{n-1} \left(B_{t_k}^8 \right)^{(3)} (x)\, dx. \quad (52)$$

Since $\left(B_{t_k}^8 \right)^{(n)}$ is the spline of order $8 - n$ on the knots of multiplicity at most two in points 0 and 1, we have

$$\left(B_{t_k}^8 \right)^{(n)} (0) = \left(B_{t_k}^8 \right)^{(n)} (1) = 0, \quad n = 0, \dots 4, \quad (53)$$

and thus

$$\int_0^1 (B_{t_k}^8)^{(4)} (x)\, dx = 0 \quad (54)$$

and

$$\int_0^1 x^n (B_{t_k}^8)^{(4)} (x)\, dx = - \int_0^1 n x^{n-1} \left(B_{t_k}^8 \right)^{(3)} (x)\, dx, \quad n = 1, \dots, 3. \quad (55)$$

Using (53) and the integration by parts three times, we obtain:

$$\int_0^1 x^n (B_{t_k}^8)^{(4)} (x)\, dx = (-1)^n\, n! \left[\left(B_{t_k}^8 \right)^{(4-n)} (1) - \left(B_{t_k}^8 \right)^{(4-n)} (0) \right] = 0, \quad (56)$$

for $n = 1, \dots, 3$, which proves the assertion. □

Remark 2. In some constructions, the condition number of the wavelet basis is improved by orthogonalization of boundary wavelets or by the orthogonalization of scaling functions on the coarsest level. In our case, the improvement was insignificant.

5.2. Norm equivalences

It remains to prove that Ψ and $\tilde{\Psi}$ are Riesz bases for the space $L^2([0,1])$ and that properly normalized basis Ψ is a Riesz basis for Sobolev space $H^s([0,1])$ for some s specified below. The proofs are based on the theory developed in [13] and [17].

For a function f defined on the real line a Sobolev exponent of smoothness is defined as $\sup\{s : f \in H^s(\mathbb{R})\}$. It is known that primal scaling functions extended to the real line by zero have the Sobolev regularity at least $\gamma = \frac{5}{2}$ and that dual scaling functions extended to the real line by zero have the Sobolev regularity at least $\tilde{\gamma} = 0.344$.

Theorem 6. *i) The sets* $\{\Phi_j\} := \{\Phi_j\}_{j \geq j_0}$ *and* $\left\{\tilde{\Phi}_j\right\} := \left\{\tilde{\Phi}_j\right\}_{j \geq j_0}$ *are uniformly stable, i.e.*

$$c\,\|b\|_{l_2(\mathcal{I}_j)} \leq \left\|\sum_{k \in \mathcal{I}_j} b_k \phi_{j,k}\right\| \leq C\,\|b\|_{l_2(\mathcal{I}_j)} \quad \text{for all } b = \{b_k\}_{k \in \mathcal{I}_j} \in l^2(\mathcal{I}_j),\ j \geq j_0. \tag{57}$$

ii) For all $j \geq j_0$*, the Jackson inequalities hold, i.e.*

$$\inf_{v_j \in S_j} \|v - v_j\| \lesssim 2^{-sj}\,\|v\|_{H^s([0,1])} \quad \text{for all } v \in H^s([0,1]) \text{ and } s < N, \tag{58}$$

and

$$\inf_{v_j \in \tilde{S}_j} \|v - v_j\| \lesssim 2^{-sj}\,\|v\|_{H^s([0,1])} \quad \text{for all } v \in H^s([0,1]) \text{ and } s < \tilde{N}. \tag{59}$$

iii) For all $j \geq j_0$*, the Bernstein inequalities hold, i.e.*

$$\|v_j\|_{H^s([0,1])} \lesssim 2^{sj}\,\|v_j\| \quad \text{for all } v_j \in S_j \text{ and } s < \gamma, \tag{60}$$

and

$$\|v_j\|_{H^s([0,1])} \lesssim 2^{sj}\,\|v_j\| \quad \text{for all } v_j \in \tilde{S}_j \text{ and } s < \tilde{\gamma}. \tag{61}$$

Proof. i) Due to Lemma 2.1 in [17], the collections $\{\Phi_j\} := \{\Phi_j\}_{j \geq j_0}$ and $\left\{\tilde{\Phi}_j\right\} := \left\{\tilde{\Phi}_j\right\}_{j \geq j_0}$ are uniformly stable, if Φ_j and $\tilde{\Phi}_j$ are biorthogonal,

$$\|\phi_{j,k}\| \lesssim 1, \left\|\tilde{\phi}_{j,k}\right\| \lesssim 1, \quad k \in \mathcal{I}_j,\ j \geq j_0, \tag{62}$$

and Φ_j and $\tilde{\Phi}_j$ are locally finite, i.e.

$$\#\left\{k' \in \mathcal{I}_j : \Omega_{j,k'} \cap \Omega_{j,k} \neq \emptyset\right\} \lesssim 1, \quad \text{for all } k \in \mathcal{I}_j,\ j \geq j_0, \tag{63}$$

and

$$\#\left\{k' \in \mathcal{I}_j : \tilde{\Omega}_{j,k'} \cap \tilde{\Omega}_{j,k} \neq \emptyset\right\} \lesssim 1, \quad \text{for all } k \in \mathcal{I}_j,\ j \geq j_0, \tag{64}$$

where $\Omega_{j,k} := \text{supp } \phi_{j,k}$ and $\tilde{\Omega}_{j,k} := \text{supp } \tilde{\phi}_{j,k}$. By (23) the sets Φ_j and $\tilde{\Phi}_j$ are biorthogonal. The properties (62), (63), and (64) follow from (9), (11), and (18).

ii) By Lemma 2.1 in [17], the Jackson inequalities are the consequences of i) and the polynomial exactness of primal and dual multiresolution analyses.

iii) The Bernstein inequalities follow from i) and the regularity of basis functions, for details see [17].

□

The following fact follows from [13].

Corollary 1. *We have the norm equivalences*

$$\|v\|_{H^s}^2 \sim 2^{2sj_0} \left\| \sum_{k \in \mathcal{I}_{j_0}} \left\langle v, \tilde{\phi}_{j_0,k} \right\rangle \phi_{j_0,k} \right\|^2 + \sum_{j=j_0}^{\infty} 2^{2sj} \left\| \sum_{k \in \mathcal{J}_j} \left\langle v, \tilde{\psi}_{j,k} \right\rangle \psi_{j,k} \right\|^2, \tag{65}$$

where $v \in H^s([0,1])$ *and* $s \in (-\tilde{\gamma}, \gamma)$.

The norm equivalence for $s = 0$, Theorem 2, and Lemma 4, imply that

$$\Psi := \Phi_{j_0} \cup \bigcup_{j=j_0}^{\infty} \Psi_j \quad \text{and} \quad \tilde{\Psi} := \tilde{\Phi}_{j_0} \cup \bigcup_{j=j_0}^{\infty} \tilde{\Psi}_j \tag{66}$$

are biorthogonal Riesz bases of the space $L^2([0,1])$. Let us define

$$\mathbf{D} = \left(\mathbf{D}_{\lambda,\tilde{\lambda}}\right)_{\lambda,\tilde{\lambda} \in \mathcal{J}}, \quad \mathbf{D}_{\lambda,\tilde{\lambda}} := \delta_{\lambda,\tilde{\lambda}} 2^{|\lambda|}, \quad \lambda, \tilde{\lambda} \in \mathcal{J}. \tag{67}$$

The relation (65) implies that $\mathbf{D}^{-s}\Psi$ is a Riesz basis of the Sobolev space $H^s([0,1])$ for $s \in (-\tilde{\gamma}, \gamma)$.

6. Quantitative properties of constructed bases

In this section, we compare quantitative properties of bases constructed in this paper, cubic spline-wavelet basis from [26] and cubic spline multiwavelet basis recently adapted to homogeneous boundary conditions in [28]. The condition of multi-scale wavelet bases is shown in Table 1. Our wavelet basis is denoted by CF, a basis from [28] is denoted by Schneider and a basis from [26] adapted to complementary boundary conditions by method from [18] is denoted by Primbs. The last basis is the same as the basis from [5] adapted to complementary boundary conditions by method from [18], see Remark 1.

Other criteria for the effectiveness of wavelet bases is the condition number of a corresponding stiffness matrix. Here, let us consider the stiffness matrix:

$$\mathbf{A}_{j_0,s} = \left(\left\langle \psi''_{j,k}, \psi''_{l,m} \right\rangle\right)_{\psi_{j,k},\psi_{l,m} \in \Psi_{j_0,s}}. \tag{68}$$

It is well-known that the condition number of $\mathbf{A}_{j_0,s}$ increases quadratically with the matrix size. To remedy this, we use a diagonal matrix for preconditioning

$$\mathbf{A}^{prec}_{j_0,s} = \mathbf{D}^{-1}_{j_0,s} \mathbf{A}_{j_0,s} \mathbf{D}^{-1}_{j_0,s}, \tag{69}$$

where

$$\mathbf{D}_{j_0,s} = \operatorname{diag}\left(\left\langle \psi''_{j,k}, \psi''_{j,k} \right\rangle^{1/2}\right)_{\psi_{j,k} \in \Psi_{j_0,s}}. \tag{70}$$

In [7] the anisotropic wavelet basis were used for solving fourth-order problems. Here, we use isotropic wavelet basis, i.e. we define multiscale wavelet basis on the unit square by

$$\Psi^{2D}_{3,s} = \Phi^{2D}_{3} \cup \bigcup_{j=3}^{s} \Psi^{2D}_{j}, \tag{71}$$

Table 1: The condition numbers of wavelet bases and stiffness matrices, $j_0 = 3$ for CF and Schneider, $j_0 = 4$ for Primbs.

	$\Psi_{j_0,j}$			$\mathbf{A}^{prec}_{j_0,j}$		
j	CF	Schneider	Primbs	CF	Schneider	Primbs
1	8.3	1.9	14.9	64.8	472.0	1111.0
3	12.5	2.4	45.9	66.5	569.5	1116.9
5	15.3	2.6	69.8	66.6	640.8	1117.0
7	18.0	2.7	85.8	66.7	693.0	1117.0

where

$$\Phi_3^{2D} = \Phi_3 \otimes \Phi_3, \quad \Psi_j^{2D} = \Phi_j \otimes \Psi_j \cup \Psi_j \otimes \Phi_j \cup \Psi_j \otimes \Psi_j. \tag{72}$$

The symbol $\otimes$ denotes the tensor product. The preconditioned stiffness matrix ${}^{2D}\mathbf{A}_{j_0,s}^{prec}$ for the biharmonic equation defined on the unit square is similar to the one dimensional case. Condition numbers of the stiffness matrices are listed in Table 1 and Table 2. The condition number of the stiffness matrix corresponding to wavelet basis by Primbs exceeds 10^4 already for number of levels $j = 3$. Wavelet basis from [17] adapted to complementary boundary conditions by method from [18] is very badly conditioned, its quantitative properties can be found in [28].

7. Numerical example

Now, we compare the quantitative behaviour of the adaptive wavelet method with our bases and bases from [28]. Both bases are formed by cubic splines and have local duals. We consider the equation

$$\Delta^2 u = f \quad \text{in} \quad \Omega, \quad u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \tag{73}$$

for $\Omega = (0,1)^2$, where the solution u is given by

$$u(x,y) = v(x)\, v(y)\,, \quad v(x) := x^2 \left(1 - \frac{e^{10x}}{e^{10}}\right)^2. \tag{74}$$

Note that the solution exhibits a sharp gradient near the point $[1,1]$. We solve the problem by the method designed in [12] with the approximate

Table 2: The condition of numbers of stiffness matrices of the size $N \times N$ for j levels.

j	N	CF	N	Schneider
1	289	128.05	900	484.35
2	1089	141.28	3844	583.41
3	4225	212.01	15876	626.91
4	16641	257.56	64516	653.45
5	66049	281.21	260100	673.19
6	263169	297.23	1044484	689.43
7	1050625	306.12	4186116	703.42

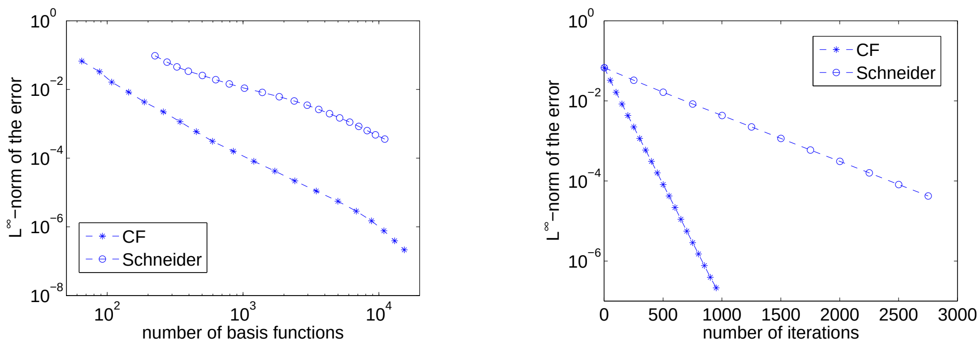

Figure 4: The convergence history for adaptive wavelet scheme with various wavelet bases.

multiplication of the stiffness matrix with a vector proposed in [6]. We use wavelets up to the scale $|\lambda| \leq 10$. The convergence history is shown in Figure 4. In our experiments, the convergence rate, i.e. the slope of the curve, is similar for both bases. However, they significantly differ in the number of basis functions and number of iterations needed to resolve the problem with desired accuracy. The number of basis functions was about 10^4 for an error in L^∞-norm about 10^{-7}. The number of all basis functions for full grid, i.e. basis functions on the level ten or less, is about 10^6, therefore by using an adaptive method the significant compression was achieved. It can seem that the number of iterations is quite large, but one could take into account that in the beginning the iterations were done for much smaller vector and the size of the vector increases successively. The algorithm is asymptotically optimal, i.e. the computational time depends linearly on the number of basis functions, see [12].

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