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Quadratic spline wavelets with short support for fourth-order problems

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Abstract In the paper, we propose constructions of new quadratic spline-wavelet bases on the interval and the unit square satisfying homogeneous Dirichlet boundary conditions of the second order. The basis functions have small supports and wavelets have one vanishing moment. We show that stiffness matrices arising from discretization of the biharmonic problem using a constructed wavelet basis have uniformly bounded condition numbers and these condition numbers are very small.

Keywords Wavelet · Quadratic spline · Homogeneous Dirichlet boundary conditions · Condition number · Biharmonic equation

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1 Introduction

In this paper, we propose a construction of quadratic spline wavelet bases on the interval that are well-conditioned, adapted to homogeneous Dirichlet boundary conditions of the second order, the wavelets have one vanishing moment and the shortest possible support. The wavelet basis of the space $H_0^2\left((0,1)^2\right)$ is then obtained by an isotropic tensor product.

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Wavelet bases are useful for solving the fourth-order problems. In [11], a construction of cubic spline wavelet basis was proposed and it was shown that the Galerkin method based on this wavelet basis is very efficient even in comparison with multigrid methods. We show that our wavelet basis is even better conditioned than basis in [11]. Moreover, since our wavelets have vanishing moments, they can be used in adaptive wavelet methods.

First of all, we summarize the desired properties of a constructed basis:

- *Riesz basis property.* We construct Riesz bases of the space $H_0^2(0,1)$ and $H_0^2\left((0,1)^2\right)$.
- *Polynomial exactness.* Since the primal basis functions are quadratic B-splines, the primal multiresolution analysis has polynomial exactness of order three.
- *Vanishing moments.* The inner wavelets have one vanishing moment, the wavelets near the boundary do not need to have vanishing moments.
- *Short support.* The wavelets have the shortest possible support among quadratic spline wavelets with one vanishing moment.
- *Locality.* The primal basis functions are local.
- *Closed form.* The primal scaling functions and wavelets are known in the closed form.
- *Homogeneous Dirichlet boundary conditions.* Our wavelet bases satisfy homogeneous Dirichlet boundary conditions of second order.
- *Well-conditioned bases.* Our objective is to construct a well conditioned wavelet basis.

Moreover, in a comparison with constructions in [2], [8], [12], [13] that are quite long and technical, the construction in this paper is very simple. Many constructions of spline wavelet or multiwavelet bases on the interval have been proposed in recent years. In [1], [2], [8], [12] cubic spline wavelets on the interval were constructed. In [7] cubic spline multiwavelet bases were designed and they were adapted to complementary boundary conditions of second order in [13]. In these cases dual functions are known and are local. Spline wavelet or multiwavelet bases whose duals are not local were constructed in [4], [9], [10], [11]. Some of these bases were already adapted to boundary conditions. The advantage of our construction is the shortest possible support for a given number of required vanishing moments. Vanishing moments are necessary in some applications such as adaptive wavelet methods [5], [6]. Originally, these methods were designed for wavelet bases with local duals. However, it was shown in [14] that wavelet bases without local dual basis can be used if the solved equation is linear.

2 Wavelet bases

This section provides a short introduction to the concept of wavelet bases in Sobolev spaces. In this paper, we consider the domain $\Omega = (0,1)$ or $\Omega = (0,1)^2$. We consider the Sobolev space or its subspace by $H \subset H^s(\Omega)$ for nonnegative integer s and the corresponding inner product by $\langle \cdot, \cdot \rangle_H$, a norm by $\|\cdot\|_H$ and a seminorm by $|\cdot|_H$. In case $s = 0$ we consider the space $L^2(\Omega)$ and we denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the L^2-inner product and the L^2-norm, respectively. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j, k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale* or a *level*. Let

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{\lambda \in \mathcal{J}} |v_\lambda|^2}, \quad \text{for} \quad \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R}, \tag{1}$$

and

$$l^2(\mathcal{J}) = \left\{ \mathbf{v} : \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R}, \|\mathbf{v}\|_2 < \infty \right\}. \tag{2}$$

A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a *(primal) wavelet basis* of H, if

i) Ψ is a *Riesz basis* for H, i.e. the closure of the span of Ψ is H and there exist constants $c, C \in (0, \infty)$ such that

$$c \|\mathbf{b}\|_2 \leq \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \leq C \|\mathbf{b}\|_2, \quad \mathbf{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}). \tag{3}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies (3)}\}$, $C_\psi := \inf\{C : C \text{ satisfies (3)}\}$ are called *Riesz bounds* and *cond* $\Psi = C_\psi / c_\psi$ is called the *condition number* of Ψ.

ii) The functions are *local* in the sense that $\operatorname{diam}(\Omega_\lambda) \leq C2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where Ω_λ is the support of ψ_λ, and at a given level j the supports of only finitely many wavelets overlap at any point $x \in \Omega$.

By the Riesz representation theorem, to any basis of the space H there exists a unique family $\tilde{\Psi} = \left\{ \tilde{\psi}_\lambda, \lambda \in \tilde{\mathcal{J}} \right\} \subset H$ biorthogonal to Ψ, i.e.

$$\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle_H = \delta_{i,j} \delta_{k,l}, \quad \text{for all} \quad (i,k) \in \mathcal{J}, \quad (j,l) \in \tilde{\mathcal{J}}, \tag{4}$$

where $\delta_{i,j}$ denotes the Kronecker delta, i.e. $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. This family is a Riesz basis for H if and only if the primal basis is a Riesz basis for H. The functions $\tilde{\psi}_{j,l}$ do not need to be local, therefore $\tilde{\Psi}$ do not need to be a wavelet basis in the sense of the above definition. The basis $\tilde{\Psi}$ is called a *dual* basis.

Wavelets are usually constructed using a function ψ called a mother-wavelet by

$$\psi_{j,k} = 2^{j/2} \psi\left(2^j x - k + n\right), n \in \mathbb{N}.$$

Also the inner wavelets in this paper are constructed by this way. This does not implicate that the dual basis has a mother-wavelet.

In many cases, the wavelet system Ψ is constructed with the aid of a multiresolution analysis. A sequence $\mathcal{V} = \{V_j\}_{j \geq j_0}$, of closed linear subspaces $V_j \subset H$ is called a *multiresolution* or *multiscale analysis*, if

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots H \tag{5}$$

and $\cup_{j \geq j_0} V_j$ is complete in H.

The nestedness and the closedness of the multiresolution analysis implies the existence of the *complement spaces* W_j such that $V_{j+1} = V_j \oplus W_j$.

We now assume that V_j and W_j are spanned by sets of basis functions

$$\Phi_j = \{\phi_{j,k}, k \in \mathcal{I}_j\}, \quad \Psi_j = \{\psi_{j,k}, k \in \mathcal{J}_j\}, \tag{6}$$

where $\mathcal{I}_j$ and $\mathcal{J}_j$ are finite or at most countable index sets. We refer to $\phi_{j,k}$ as *scaling functions* and $\psi_{j,k}$ as *wavelets*. The multiscale basis and the wavelet basis of H are given by

$$\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j, \quad \Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j. \tag{7}$$

Let us denote

$$\tilde{\Phi}_j = \left\{\tilde{\phi}_{j,k}, k \in \mathcal{I}_j\right\}, \quad \tilde{\Psi}_j = \left\{\tilde{\psi}_{j,k}, k \in \mathcal{J}_j\right\}, \tag{8}$$

and

$$\tilde{V}_j = \mathrm{span}\, \tilde{\Phi}_j, \quad \tilde{W}_j = \mathrm{span}\, \tilde{\Psi}_{\mathrm{j}}. \tag{9}$$

The spaces $\tilde{V}_j$ are also nested:

$$\tilde{V}_j \subset \tilde{V}_{j+1}, \quad j \geq j_0. \tag{10}$$

Most common way of construction of wavelet bases is using dual functions. In our paper, we use a different approach and construct scaling functions $\phi_{j,k}$ as quadratic splines and we derive wavelets $\psi_{j,k}$ directly as linear combinations of functions $\phi_{j+1,k}$, where the coefficients of the linear combinations are chosen such that wavelets have vanishing moments.

Polynomial exactness of order $N \in \mathbb{N}$ for the primal scaling basis and of order $\tilde{N} \in \mathbb{N}$ for the dual scaling basis is another desired property of wavelet bases. It means that

$$\mathbb{P}_{N-1}(\Omega) \subset V_j, \quad \mathbb{P}_{\tilde{N}-1}(\Omega) \subset \tilde{V}_j, \quad j \geq j_0, \tag{11}$$

where $\mathbb{P}_m(\Omega)$ is the space of all algebraic polynomials on Ω of degree less or equal to m.

The polynomial exactness of order $\tilde{N}$ on the dual side is equivalent to $\tilde{N}$ vanishing wavelet moments on the primal side, i.e.

$$\int_\Omega P(x)\, \psi_\lambda(x)\, dx = 0, \quad \text{for any } P \in \mathbb{P}_{\tilde{N}-1},\ \psi_\lambda \in \bigcup_{j \geq j_0} \Psi_j. \tag{12}$$

3 Primal scaling basis

A primal scaling basis is generated from function ϕ. Let ϕ be a quadratic B-spline defined on knots $[0,1,2,3]$. It can be written explicitly as:

$$\phi(x)=\begin{cases}\frac{x^2}{2}, & x\in[0,1],\\ -x^2+3x-\frac{3}{2}, & x\in[1,2],\\ \frac{x^2}{2}-3x+\frac{9}{2}, & x\in[2,3],\\ 0, & \text{otherwise},\end{cases} \tag{13}$$

The function ϕ satisfies a scaling equation [8]:

$$\phi(x)=\frac{\phi(2x)}{4}+\frac{3\phi(2x-1)}{4}+\frac{3\phi(2x-2)}{4}+\frac{\phi(2x-3)}{4}. \tag{14}$$

For $j\geq 2$ and $x\in[0,1]$ we set

$$\phi_{j,k}(x)=2^{j/2}\phi(2^jx-k+1),\ k=1,\ldots 2^j-2. \tag{15}$$

The graphs of the functions $\phi_{j,k}$ on the coarsest level $j=2$ are displayed in Figure 1.

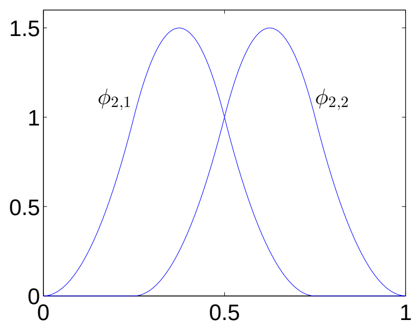

Fig. 1 Primal scaling basis for $j=2$.

We define a wavelet ψ as

$$\psi(x)=-\frac{1}{2}\phi(2x-1)+\frac{1}{2}\phi(2x-2). \tag{16}$$

Then $\operatorname{supp}\psi=[0.5,2.5]$ and ψ has one vanishing wavelet moments, i.e.

$$\int_{-\infty}^{\infty}\psi(x)dx=0. \tag{17}$$

The graph of ψ is shown in Figure 2.

We define a boundary wavelet ψ_b by:

$$\psi_b(x)=a\,\phi(2x)+b\,\phi(2x-1), \tag{18}$$

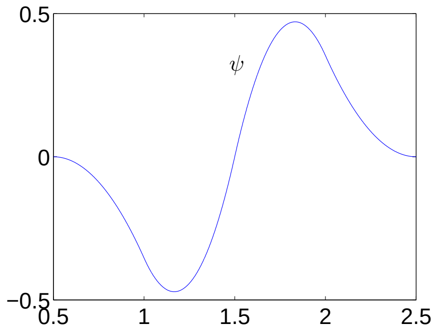

Fig. 2 Wavelet ψ.

where a and b are real parameters. Since we want to have wavelets with the shortest possible support for a given number of vanishing moments, we will consider two choices of the parameters:

a) $a = -\frac{1}{2}$, $b = \frac{1}{2}$,
b) $a = 1$, $b = 0$.

The properties of these wavelets are summarized in the following lemma.

Lemma 1 *i) The function* $\psi_b(x)$ *defined by (18) with the choice of parameters a) satisfies* $\operatorname{supp} \psi_b = [0, 2]$ *and*

$$\int_{-\infty}^{\infty} \psi_b(x) dx = 0. \tag{19}$$

ii) The function $\psi_b(x)$ *defined by (18) with the choice of parameters b) satisfies* $\operatorname{supp} \psi_b = \left[0, \frac{3}{2}\right]$.

Proof The length of the support of the function ψ_b is derived from the lengths of the supports of the functions $\phi(2x)$ and $\phi(2x-1)$. By (13) we have

$$\operatorname{supp} \phi(2x) = [0, 1.5] \quad \text{and } \operatorname{supp} \phi(2x-1) = [0.5, 2]. \tag{20}$$

Since the functions $\phi(2x)$ and $\phi(2x-1)$ are given in the closed form, the formula (19) can be verified easily.

Thus, we can choose boundary wavelet with one vanishing moment and larger support or boundary wavelets with shorter supports but without vanishing moments. If $f \in H_0^2(0,1)$ and f is constant at the interval $[0, \epsilon]$, $0 < \epsilon < 1$, then f has to be zero at $[0, \epsilon]$. The same holds for the interval $[1-\epsilon, 1]$. Hence $f \in H_0^2(0,1)$ can not be nonzero constant near the boundary and therefore in some applications such as adaptive wavelet methods the vanishing moment does not play the significant role for boundary wavelets. The graphs of boundary wavelets ψ_b are displayed in Figure 3. All the following lemmas and theorems are valid for both choices of parameters.

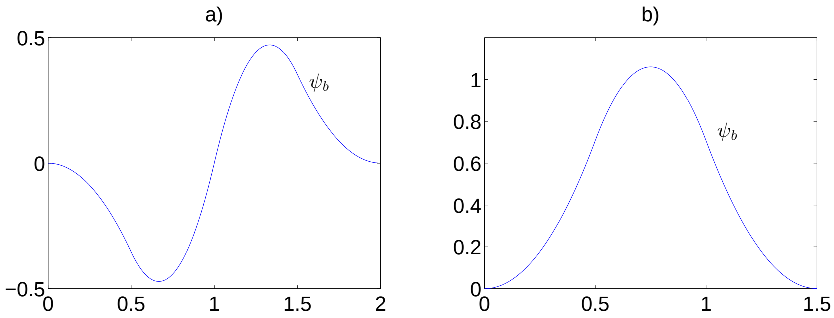

Fig. 3 Boundary wavelet ψ_b for a) and b), respectively.

For $j \geq 2$ and $x \in [0,1]$ we define

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k + 2), k = 2, ..., 2^j - 1, \qquad (21)$$
$$\psi_{j,1}(x) = 2^{j/2}\psi_b(2^j x), \quad \psi_{j,2^j}(x) = 2^{j/2}\psi_b(2^j(1-x)).$$

We denote

$$\Phi_j = \left\{\phi_{j,k}/\left|\phi_{j,k}\right|_{H_0^2(0,1)}, k = 1, \ldots, 2^j - 2\right\}, \qquad (22)$$
$$\Psi_j = \left\{\psi_{j,k}/\left|\psi_{j,k}\right|_{H_0^2(0,1)}, k = 1, \ldots, 2^j\right\}.$$

In Section 5 we show that the sets

$$\Psi^s = \Phi_2 \cup \bigcup_{j=2}^{1+s} \Psi_j \quad \text{and} \quad \Psi = \Phi_2 \cup \bigcup_{j=2}^{\infty} \Psi_j \qquad (23)$$

are a multiscale wavelet basis and a wavelet basis of the space $H_0^2(0,1)$, respectively. We use $u \otimes v$ to denote the tensor product of functions u and v, i.e. $(u \otimes v)(x_1, x_2) = u(x_1)\, v(x_2)$. We set

$$F_j = \left\{\phi_{j,k} \otimes \phi_{j,l} \,/\, \left|\phi_{j,k} \otimes \phi_{j,l}\right|_{H_0^2(\Omega)}, k, l = 1, \ldots, 2^j - 2\right\}$$
$$G_j^1 = \left\{\phi_{j,k} \otimes \psi_{j,l} \,/\, \left|\phi_{j,k} \otimes \psi_{j,l}\right|_{H_0^2(\Omega)}, k = 1, \ldots, 2^j - 2, l = 1, \ldots 2^j\right\}$$
$$G_j^2 = \left\{\psi_{j,k} \otimes \phi_{j,l} \,/\, \left|\psi_{j,k} \otimes \phi_{j,l}\right|_{H_0^2(\Omega)}, k = 1, \ldots, 2^j, l = 1, \ldots 2^j - 2\right\}$$
$$G_j^3 = \left\{\psi_{j,k} \otimes \psi_{j,l} \,/\, \left|\psi_{j,k} \otimes \psi_{j,l}\right|_{H_0^2(\Omega)}, k, l = 1, \ldots, 2^j\right\}$$

where $\Omega = (0,1)^2$. We show that the sets defined by

$$\Psi_s^{2D} = F_2 \cup \bigcup_{j=2}^{1+s} \left(G_j^1 \cup G_j^2 \cup G_j^3\right), \quad \Psi^{2D} = F_2 \cup \bigcup_{j=2}^{\infty} \left(G_j^1 \cup G_j^2 \cup G_j^3\right) \qquad (24)$$

are a wavelet basis and a multiscale wavelet basis of the space $H_0^2(\Omega)$.

4 Refinement matrices

From the nestedness and the closedness of multiresolution spaces it follows that there exist *refinement matrices* $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ such that

$$\Phi_j = \mathbf{M}_{j,0}^T \Phi_{j+1}, \quad \Psi_j = \mathbf{M}_{j,1}^T \Phi_{j+1}. \tag{25}$$

By (14), the entries of the refinement matrix $\mathbf{M}_{j,0}$ satisfy:

$$\left(\mathbf{M}_{j,0}\right)_{m,n} = \begin{cases} \frac{h_{m+2-2n}}{\sqrt{2}}, & n = 1, \dots, 2^j - 2, \ 1 \le m+2-2n \le 4, \\ 0, & \text{otherwise}, \end{cases} \tag{26}$$

where

$$\mathbf{h} = [h_1, h_2, h_3, h_4] = \left[\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right] \tag{27}$$

is a vector of coefficients from scaling equation (14).

It follows from the equations (16) and (18) that the matrix $\mathbf{M}_{j,1}$ is of the size $\left(2^{j+1} - 2\right) \times 2^j$ and has the structure

$$\mathbf{M}_{j,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} a & b & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & & 0 & 0 \\ \vdots & \vdots & & & & & & & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & b & a \end{pmatrix}^T \tag{28}$$

There also exist refinement matrices $\tilde{\mathbf{M}}_{j,0}$ and $\tilde{\mathbf{M}}_{j,1}$ corresponding to dual spaces that satisfy:

$$\tilde{\Phi}_j = \tilde{\mathbf{M}}_{j,0}^T \tilde{\Phi}_{j+1}, \quad \tilde{\Psi}_j = \tilde{\mathbf{M}}_{j,1}^T \tilde{\Phi}_{j+1}. \tag{29}$$

The structure of the matrix $\tilde{\mathbf{M}}_{j,0}$ is derived in the proof of Lemma 2. We do not need to know the structure of the matrix $\tilde{\mathbf{M}}_{j,1}$ in this paper.

The Euclidean norm of a vector $\mathbf{v}$ is denoted by $\|\mathbf{v}\|_2$ and the spectral norm of the matrix $\mathbf{M}$ is denoted as $\|\mathbf{M}\|_2$. The following lemma is crucial for the proof of a Riesz basis property.

Lemma 2 *The norm of the matrix* $\tilde{\mathbf{M}}_{j,0}$ *satisfies* $\left\|\tilde{\mathbf{M}}_{j,0}\right\|_2 \le 2^p, p = \frac{\ln 6}{\ln 4}$.

Proof We prove the lemma for the choice a) of parameters for the boundary wavelet, for the choice b) the proof is similar. We denote the entries of the matrix $\tilde{\mathbf{M}}_{j,0}$ as $\tilde{M}_{k,l}$, $k = 1, \dots 2^{j+1} - 2$, $l = 1, \dots, 2^j - 2$.

Due to the biorthogonality of the sets $\Psi_j \cup \Phi_j$ and $\tilde{\Psi}_j \cup \tilde{\Phi}_j$ we have

$$\mathbf{M}_{j,0}^T \tilde{\mathbf{M}}_{j,0} = \mathbf{I}_j \tag{30}$$

and

$$\mathbf{M}_{j,1}^T \tilde{\mathbf{M}}_{j,0} = \mathbf{0}_j, \tag{31}$$

where $\mathbf{I}_j$ denotes the identity matrix and $\mathbf{0}_j$ denotes the zero matrix of the appropriate size.

From (28) and (31) we have for $l = 1, \ldots, 2^j - 2$:

$$\tilde{M}_{1,l} = \tilde{M}_{2,l}, \quad \tilde{M}_{2^{j+1}-2,l} = \tilde{M}_{2^{j+1}-3,l}, \tag{32}$$

and

$$\tilde{M}_{k,l} = \tilde{M}_{k+1,l}, \quad \text{for } k \text{ even}, \quad k = 2, \ldots, 2^{j+1} - 4. \tag{33}$$

We substitute these relations into (30) and we obtain a new system of equations $\mathbf{A}_j \mathbf{B}_j = \mathbf{I}_j$, where

$$\mathbf{A}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{7}{4} & \frac{1}{4} & 0 & \ldots & & 0 \\ \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & \vdots \\ 0 & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & & \ldots & 0 & \frac{1}{4} & \frac{7}{4} \end{pmatrix} \tag{34}$$

and $\mathbf{B}_j$ contains $\tilde{M}_{k,l}$ for k even, i.e. the entries $B_{k,l}$ of the matrix $\mathbf{B}_j$ satisfy:

$$B_{k,l} = \tilde{M}_{2k,l}, \quad k, l = 1, \ldots 2^j - 2. \tag{35}$$

We factorize the matrix $\mathbf{A}_j$ as $\mathbf{A}_j = \mathbf{C}_j \mathbf{D}_j$, where

$$\mathbf{C}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{3+2\sqrt{2}}{4} & \frac{1}{4} & 0 & 0 & \ldots & 0 \\ \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & \vdots \\ 0 & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & \ldots & 0 & 0 & \frac{1}{4} & \frac{3+2\sqrt{2}}{4} \end{pmatrix} \tag{36}$$

and

$$\mathbf{D}_j = \begin{pmatrix} \frac{7-a}{3+2\sqrt{2}} & 0\ 0 \ldots 0\ 0 & \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-4}} \\ a & 1\ 0 \quad\ 0\ 0 & \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-5}} \\ \frac{a}{-3-2\sqrt{2}} & 0\ 1 \quad\ 0\ 0 & \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-6}} \\ \vdots & \vdots \quad \ddots \quad \vdots & \vdots \\ \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-6}} & 0\ 0 \quad\ 1\ 0 & \frac{a}{-3-2\sqrt{2}} \\ \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-5}} & 0\ 0 \quad\ 0\ 1 & a \\ \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-4}} & 0\ 0 \ldots 0\ 0 & \frac{7-a}{3+2\sqrt{2}} \end{pmatrix}, \tag{37}$$

with

$$a = \frac{1-\sqrt{2}}{6+4\sqrt{2}}. \tag{38}$$

More precisely, the entries $D_{k,l}$ of the matrix $\mathbf{D}_j$ are given by:

$$\begin{aligned} D_{1,1} &= D_{2^j-2,2^j-2} = \frac{7-a}{3+2\sqrt{2}}, \\ D_{k,1} &= D_{2^j-1-k,2^j-2} = \frac{a}{\left(-3-2\sqrt{2}\right)^{k-2}}, \quad \text{for } k = 2,\dots,2^j-2, \\ D_{k,k} &= 1, \quad \text{for } k = 2,\dots,2^j-3, \\ D_{k,l} &= 0, \quad \text{otherwise.} \end{aligned} \tag{39}$$

It is easy to verify that $\tilde{\mathbf{C}}_j = \mathbf{C}_j^{-1}$ has entries:

$$\tilde{C}_{k,l} = \frac{1}{\left(-3-2\sqrt{2}\right)^{|k-l|}}, \tag{40}$$

and the matrix $\mathbf{D}_j^{-1}$ has the structure:

$$\mathbf{D}_j^{-1} = \begin{pmatrix} d_1 & 0 \dots 0 & d_n \\ d_2 & 1 \quad\;\; 0 & d_{n-1} \\ \vdots & \ddots & \vdots \\ d_{n-1} & 0 \quad\;\; 1 & d_2 \\ d_n & 0 \dots 0 & d_1 \end{pmatrix}, \tag{41}$$

with $n = 2^j - 2$ and

$$\begin{aligned} d_1 &= \frac{\left(3+2\sqrt{2}\right)\,\alpha_n}{7-a} \\ d_k &= \frac{a\,\alpha_n}{(7-a)\left(-3-2\sqrt{2}\right)^{k-3}} - \frac{\alpha_n\,\beta_n}{\left(-3-2\sqrt{2}\right)^{n-k}}, \quad k = 2,\dots,n-1, \\ d_n &= \frac{-a\,\alpha_n}{(7-a)^2\,\left(-3-2\sqrt{2}\right)^{2^j-6}}, \end{aligned} \tag{42}$$

where the constants α_n and β_n are given by

$$\alpha_n = \left(1 - \frac{a^2}{(7-a)^2\left(-3-2\sqrt{2}\right)^{2n-6}}\right)^{-1} \tag{43}$$

and

$$\beta_n = \frac{a^2}{(7-a)^2\left(-3-2\sqrt{2}\right)^{n-5}}. \tag{44}$$

Note that $\alpha_n \approx 1$ and $\beta_n \approx 0$.

Since the matrices $\mathbf{C}_j$ and $\mathbf{D}_j$ are invertible, we can define $\mathbf{B}_j = \mathbf{A}_j^{-1} = \mathbf{D}_j^{-1}\mathbf{C}_j^{-1}$. Substituting this into (31) we obtain the entries of the matrix $\tilde{\mathbf{M}}_{j,0}$:

$$\tilde{M}_{1,l} = \frac{d_1}{\left(-3-2\sqrt{2}\right)^{|1-l|}} + \frac{d_n}{\left(-3-2\sqrt{2}\right)^{|n-l|}}, \tag{45}$$

$$\tilde{M}_{1,2^j-l} = \tilde{M}_{2,2^j-l} = \tilde{M}_{3,2^j-l} = \tilde{M}_{2^{j+1}-4,l} = \tilde{M}_{2^{j+1}-3,l} = \tilde{M}_{2^{j+1}-2,l},$$

and for $k = 1,\ldots,2^j-2$, $l = 1,\ldots,2^j-2$, we have

$$\tilde{M}_{2k,l} = B_{k,l} = \frac{1}{\left(-3-2\sqrt{2}\right)^{|k-l|}} + \frac{d_k}{\left(-3-2\sqrt{2}\right)^{|1-l|}} + \frac{d_{n+1-k}}{\left(-3-2\sqrt{2}\right)^{|n-l|}} \tag{46}$$

The entries $\tilde{M}_{2k-1,l}$ are given by (33).

It is well-known that for any matrix $\mathbf{M}$ of the size $m \times n$ with entries $M_{k,l}$:

$$\|\mathbf{M}\|_2 \le \sqrt{\|\mathbf{M}\|_1 \, \|\mathbf{M}\|_\infty}, \tag{47}$$

where

$$\|\mathbf{M}\|_1 = \max_{l=1,\ldots,n} \sum_{k=1}^{m} |M_{k,l}|, \quad \|\mathbf{M}\|_\infty = \max_{k=1,\ldots,m} \sum_{l=1}^{n} |M_{k,l}|. \tag{48}$$

In our case, from (45), (48), and a formula for a sum of a geometric sequence we obtain:

$$\left\|\tilde{\mathbf{M}}_{j,0}\right\|_1 \le 3\sqrt{2} \quad \text{and} \quad \left\|\tilde{\mathbf{M}}_{j,0}\right\|_\infty \le \sqrt{2}. \tag{49}$$

Thus

$$\left\|\tilde{\mathbf{M}}_{j,0}\right\|_2 \le \sqrt{6} = 2^p \quad \text{for} \quad p = \frac{\ln 6}{\ln 4}. \tag{50}$$

5 Riesz basis on Sobolev space

In this section, we show that Ψ and Ψ^{2D} are Riesz bases. We use Theorem 5.3. from [11]. It says that if P_j is a linear projection from V_{j+1} onto V_j and for $0 < p < q$ there exists a constant C such that

$$\|P_m P_{m+1} \ldots P_{n-1}\| \le C\, 2^{p\,(n-m)}, \tag{51}$$

then

$$\left\{2^{-4}\phi_{2,k}, k = 1, 2\right\} \cup \left\{2^{-2j}\psi_{j,k}, j \ge 2, k = 1, \ldots 2^j\right\} \tag{52}$$

is a Riesz basis of $H_0^q(0,1)$.

First we define suitable projections P_j from V_{j+1} onto V_j and show that these projections satisfies (51). Then we show that Ψ which differs from (52) only by scaling is also a Riesz basis of $H_0^2(0,1)$. We denote

$$\mathcal{I}_j = \left\{1, 2, \ldots, 2^j - 2\right\} \quad \text{and} \quad \mathcal{J}_j = \left\{1, 2, \ldots, 2^j\right\} \tag{53}$$

and for $j \geq 2$ we define

$$\Gamma_j = \{\phi_{j,k}\}_{k \in \mathcal{I}_j} \cup \{\psi_{j,k}\}_{k \in \mathcal{J}_j} \quad \text{and} \quad \mathbf{F}_j = \langle \Gamma_j, \Gamma_j \rangle . \tag{54}$$

Let a set

$$\hat{\Gamma}_j = \left\{ \hat{\phi}_{j,k} \right\}_{k \in \mathcal{I}_j} \cup \left\{ \hat{\psi}_{j,k} \right\}_{k \in \mathcal{J}_j} \tag{55}$$

be given by

$$\hat{\Gamma}_j = \mathbf{F}_j^{-1} \Gamma_j . \tag{56}$$

Since obviously

$$\left\langle \Gamma_j, \hat{\Gamma}_j \right\rangle = \mathbf{I}_j , \tag{57}$$

functions from $\hat{\Gamma}_j$ are duals to functions from Γ_j in the space V_{j+1}. Since $\mathbf{F}_j^{-1}$ is not a sparse matrix, these duals are not local. We define a projection P_j from V_{j+1} onto V_j by

$$P_j f = \sum_{k \in \mathcal{I}_j} \left\langle f, \hat{\phi}_{j,k} \right\rangle \phi_{j,k} . \tag{58}$$

Lemma 3 *Let* $f \in V_{j+1}$, $a_k^j = \left\langle f, \hat{\phi}_{j,k} \right\rangle$, $\mathbf{a}_j = \left\{ a_k^j \right\}_{k \in \mathcal{I}_j}$, $j \geq 2$, *and* $\mathbf{S}_j : \mathbf{a}_{j+1} \mapsto \mathbf{a}_j$. *Then* $\|\mathbf{S}_j\|_2 \leq 2^p$, $p = \frac{\ln 6}{\ln 4}$.

Proof We have

$$\begin{aligned} P_j f &= \sum_{k \in \mathcal{I}_j} a_k^j \phi_{j,k} = \sum_{k \in \mathcal{I}_j} \left\langle f, \hat{\phi}_{j,k} \right\rangle \phi_{j,k} \\ &= \sum_{k \in \mathcal{I}_j} \sum_{l \in \mathcal{I}_{j+1}} a_l^{j+1} \left\langle \phi_{j+1,l}, \hat{\phi}_{j,k} \right\rangle \phi_{j,k} . \end{aligned} \tag{59}$$

Therefore

$$a_k^j = \sum_{l \in \mathcal{I}_{j+1}} a_l^{j+1} \left\langle \phi_{j+1,l}, \hat{\phi}_{j,k} \right\rangle . \tag{60}$$

Let us denote

$$S_{l,k}^j = \left\langle \hat{\phi}_{j,k}, \phi_{j+1,l} \right\rangle , \quad \mathbf{S}_j = \left\{ S_{l,k}^j \right\}_{l \in \mathcal{I}_{j+1}, k \in \mathcal{I}_j} \tag{61}$$

then we can write

$$\mathbf{a}_j = \mathbf{S}_j \mathbf{a}_{j+1} , \tag{62}$$

and

$$\mathbf{S}_j = \left\langle \hat{\Phi}_j, \Phi_{j+1} \right\rangle = \left\langle \hat{\Phi}_j, \tilde{\mathbf{M}}_{j,0} \Phi_j + \tilde{\mathbf{M}}_{j,1} \Psi_j \right\rangle = \tilde{\mathbf{M}}_{j,0} . \tag{63}$$

By Lemma 2 the assertion is proved.

Lemma 4 *A projection* P_j *satisfies*

$$\|P_m P_{m+1} \dots P_{n-1}\| \leq C \, 2^{p\,(n-m)} , \quad p = \frac{\ln 6}{\ln 4} , \tag{64}$$

for all $2 \leq m < n$ *and a constant* C *independent on* m *and* n.

Proof Let $f_n \in V_n$ and $f_m = P_m P_{m+1} \dots P_{n-1} f_n$. We represent f_j by $f_j = \sum_{k \in \mathcal{I}_j} a_k^j \phi_j$ for $j = m, n$ and we set $\mathbf{a}_j = \left\{ a_k^j \right\}_{k \in \mathcal{I}_j}$. It is known [1] that $\{\phi_{j,k}, k \in \mathcal{I}_j\}$ is a Riesz basis of $V_j = \operatorname{span} \Phi_j$ and there exist constants C_1 and C_2 independent of j such that:

$$C_1 \left\| \mathbf{a}_j \right\|_2 \le \left\| \sum_{k \in \mathcal{I}_j} a_k^j \phi_{j,k} \right\| \le C_2 \left\| \mathbf{a}_j \right\|_2. \tag{65}$$

By Lemma 3 we have for $p = \frac{\ln 6}{\ln 4}$:

$$\begin{aligned} \|f_m\| &\le C_2 \left\| \mathbf{a}_m \right\|_2 \le C_2 \left\| \mathbf{S}_m \right\|_2 \left\| \mathbf{S}_{m+1} \right\|_2 \dots \left\| \mathbf{S}_{n-1} \right\|_2 \left\| \mathbf{a}_n \right\|_2 \\ &\le C_2 2^{p(n-m)} \left\| \mathbf{a}_n \right\|_2 \le C_1^{-1} C_2 2^{p(n-m)} \left\| f_n \right\|. \end{aligned} \tag{66}$$

Thus (64) is proved.

Theorem 1 *The set*

$$\left\{ 2^{-4} \phi_{2,k}, k = 1, 2 \right\} \cup \left\{ 2^{-2j} \psi_{j,k}, j \ge 2, k = 1, \dots 2^j \right\} \tag{67}$$

is a Riesz basis of $H_0^\mu(0,1)$ *for* $\frac{\ln 6}{\ln 4} < \mu < 2.5$.

Proof By Lemma 4 and Theorem 5.3. from [11], the set

$$\left\{ 2^{-4} \phi_{2,k}, k = 1, 2 \right\} \cup \left\{ 2^{-2j} \psi_{j,k}, j \ge 2, k = 1, \dots 2^j \right\} \tag{68}$$

is a Riesz basis of the space $H_0^\mu(0,1)$ for $\frac{\ln 6}{\ln 4} < \mu < \nu$, where ν is the Sobolev exponent of smoothness of the basis, i.e. $\nu = 2.5$.

Theorem 2 *The set* Ψ *is a Riesz basis of* $H_0^2(0,1)$.

Proof From (21) there exist nonzero constants C_1 and C_2 such that

$$C_1 2^{2j} \le \left| \psi_{j,k} \right|_{H_0^2(\Omega)} \le C_2 2^{2j}, \quad \text{for } j \ge 2, \quad k = 1, \dots, 2^j, \tag{69}$$

and

$$C_1 2^4 \le \left| \phi_{2,k} \right|_{H_0^2(\Omega)} \le C_2 2^4, \quad \text{for } k = 1, 2. \tag{70}$$

Let $\hat{\mathbf{b}} = \{\hat{a}_{2,k},\ k \in \mathcal{I}_2\} \cup \left\{ \hat{b}_{j,k}, j \ge 2,\ k \in \mathcal{J}_j \right\}$ be such that

$$\left\| \hat{\mathbf{b}} \right\|_2^2 = \sum_{k \in \mathcal{I}_2} \hat{a}_{2,k}^2 + \sum_{k \in \mathcal{J}_j, j \ge 2} \hat{b}_{j,k}^2 < \infty. \tag{71}$$

We define

$$a_{2,k} = \frac{2^4 \hat{a}_{2,k}}{\left| \phi_{2,k} \right|_{H_0^2(0,1)}},\ k \in \mathcal{I}_2, \quad b_{j,k} = \frac{2^{2j} \hat{b}_{j,k}}{\left| \psi_{j,k} \right|_{H_0^2(0,1)}},\ j \ge 2,\ k \in \mathcal{J}_j, \tag{72}$$

and $\mathbf{b} = \{a_{2,k},\ k \in \mathcal{I}_2\} \cup \{b_{j,k}, j \geq 2,\ k \in \mathcal{J}_j\}$. Then

$$\|\mathbf{b}\|_2 \leq \frac{\left\|\hat{\mathbf{b}}\right\|_2}{C_1} < \infty. \tag{73}$$

Theorem 1 implies that there exist constants C_3 and C_4 such that

$$C_3 \|\mathbf{b}\|_2 \leq \left\| \sum_{k \in \mathcal{I}_2} a_{2,k} 2^{-4} \phi_{2,k} + \sum_{k \in \mathcal{J}_j, j \geq 2} b_{j,k} 2^{-2j} \psi_{j,k} \right\|_{H_0^2(0,1)} \leq C_4 \|\mathbf{b}\|_2 . \tag{74}$$

Therefore

$$\begin{aligned} \frac{C_4}{C_1} \left\|\hat{\mathbf{b}}\right\|_2 \geq C_4 \|\mathbf{b}\|_2 &\geq \left\| \sum_{k \in \mathcal{I}_2} a_{2,k} 2^{-4} \phi_{2,k} + \sum_{k \in \mathcal{J}_j, j \geq 2} b_{j,k} 2^{-2j} \psi_{j,k} \right\|_{H_0^2(0,1)} \\ &= \left\| \sum_{k \in \mathcal{I}_2} \frac{\hat{a}_{2,k}}{|\phi_{2,k}|_{H_0^2(0,1)}} \phi_{2,k} + \sum_{k \in \mathcal{J}_j, j \geq 2} \frac{\hat{b}_{2,k}}{|\psi_{j,k}|_{H_0^2(0,1)}} \psi_{j,k} \right\|_{H_0^2(0,1)} \end{aligned} \tag{75}$$

and similarly

$$\frac{C_3}{C_2} \left\|\hat{\mathbf{b}}\right\|_2 \leq \left\| \sum_{k \in \mathcal{I}_2} \frac{\hat{a}_{2,k}}{|\phi_{2,k}|_{H_0^2(0,1)}} \phi_{2,k} + \sum_{k \in \mathcal{J}_j, j \geq 2} \frac{\hat{b}_{2,k}}{|\psi_{j,k}|_{H_0^2(0,1)}} \psi_{j,k} \right\|_{H_0^2(0,1)} . \tag{76}$$

Theorem 3 *The set Ψ^{2D} is a Riesz basis of $H_0^2\left((0,1)^2\right)$.*

Proof The theorem is a consequence of Lemma 1, (69), and Theorem 5.3. from [11].

6 Quantitative properties of constructed bases

In this section, we present the condition numbers of the stiffness matrices for the biharmonic problem in two dimensions. We consider the biharmonic equation

$$\Delta^2 u = f \quad \text{on} \quad \Omega = (0,1)^2, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{77}$$

where Δ is the Laplace operator. The variational formulation is $\mathbf{Au} = \mathbf{f}$, where $\mathbf{A} = \left\langle \Delta\Psi^{2D}, \Delta\Psi^{2D} \right\rangle$, $u = \mathbf{u}^T \Psi^{2D}$, and $\mathbf{f} = \left\langle f, \Psi^{2D} \right\rangle$. It is known that then $\operatorname{cond} \mathbf{A} \leq C < \infty$. Since $\mathbf{A}$ is symmetric and positive definite, we have also

$$\operatorname{cond} \mathbf{A}_s \leq C, \qquad \text{where} \;\; \mathbf{A}_s = \left\langle \Delta\Psi_s^{2D}, \Delta\Psi_s^{2D} \right\rangle. \tag{78}$$

The condition numbers of the stiffness matrices $\mathbf{A}_s$ are shown in Table 1. For the basis *b*) the condition number is even smaller than for a wavelet basis in [11].

Table 1 The condition numbers of the stiffness matrices $\mathbf{A}_s$ of the size $N \times N$ corresponding to multiscale wavelet bases with s levels of wavelets.

s	N	a)	b)
1	36	37.3	6.1
2	196	62.1	7.8
3	900	80.1	8.7
4	3844	92.3	9.8
5	15876	100.4	10.5
6	64516	106.3	11.1

7 Numerical example

We present the quantitative behaviour of the adaptive wavelet method using the bases constructed in this paper. We consider the equation (77) with a solution u given by

$$u(x,y) = v(x)\,v(y)\,, \quad v(x) = x^2\left(1 - e^{10x-10}\right)^2. \tag{79}$$

The solution exhibits a sharp gradient near the point $[1,1]$. We solve the problem by the method designed in [6] with the approximate multiplication of the stiffness matrix with a vector proposed in [3]. We use wavelets up to the scale $|\lambda| \leq 10$. The convergence history is shown in Figure 4. In our

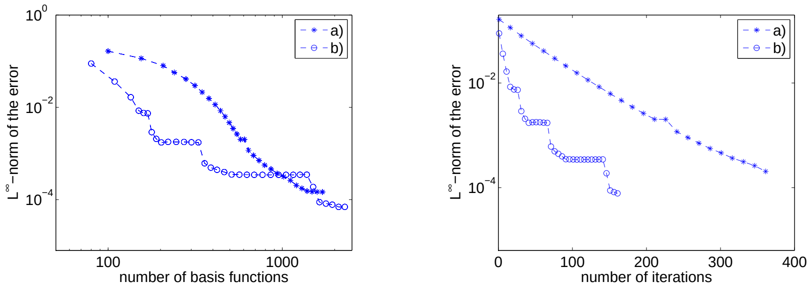

Fig. 4 The convergence history for adaptive wavelet scheme with various wavelet bases.

experiments, the convergence rate, i.e. the slope of the curve, is similar for both bases. However, bases a) and b) significantly differ in the number of iterations needed to resolve the problem with desired accuracy. The number of basis functions in both cases was about 10^3 for an error in L^∞-norm about 10^{-4}. The number of all basis functions for full grid, i.e. basis functions of the level ten or less, is about 10^6, therefore by using an adaptive method the significant compression was achieved. It can seem that the number of iterations is quite large, but one could take into account that at the beginning the iterations were done for a much smaller vector and the size of this vector increases successively.

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