Noname manuscript No.
(will be inserted by the editor)

Construction of Optimally Conditioned Cubic Spline Wavelets on the Interval

Dana Černá · **Václav Finěk**

Received: date / Accepted: date

Abstract The paper is concerned with a construction of new spline-wavelet bases on the interval. The resulting bases generate multiresolution analyses on the unit interval with the desired number of vanishing wavelet moments for primal and dual wavelets. Both primal and dual wavelets have compact support. Inner wavelets are translated and dilated versions of well-known wavelets designed by Cohen, Daubechies, and Feauveau. Our objective is to construct interval spline-wavelet bases with the condition number which is close to the condition number of the spline wavelet bases on the real line, especially in the case of the cubic spline wavelets. We show that the constructed set of functions is indeed a Riesz basis for the space $L^2([0,1])$ and for the Sobolev space $H^s([0,1])$ for a certain range of s. Then we adapt the primal bases to the homogeneous Dirichlet boundary conditions of the first order and the dual bases to the complementary boundary conditions. Quantitative properties of the constructed bases are presented. Finally, we compare the efficiency of an adaptive wavelet scheme for several spline-wavelet bases and we show a superiority of our construction. Numerical examples are presented for the one-dimensional and two-dimensional Poisson equations where the solution has steep gradients.

Keywords Biorthogonal wavelets · Interval · Spline · Condition number

Mathematics Subject Classification (2000) 65T60 · 65N99

1 Introduction

Wavelets are by now a widely accepted tool in signal and image processing as well as in numerical simulation. In the field of numerical analysis, methods based on wavelets are successfully used especially for preconditioning of large systems arising from discretization of elliptic partial differential equations, sparse representations of some types of operators and adaptive solving of operator equations. The quantitative performance of such methods strongly depends on a choice of a wavelet basis, in particular on its condition number.

Wavelet bases on a bounded domain are usually constructed in the following way: Wavelets on the real line are adapted to the interval and then by a tensor product technique to the n-dimensional cube. Finally by splitting the domain into subdomains which are images of $(0,1)^n$ under appropriate parametric mappings one can obtain wavelet bases on a fairly general domain. Thus, the properties of the employed wavelet basis on the interval are crucial for the properties of the resulting bases on general a domain.

D. Černá
Department of Mathematics and Didactics of Mathematics, Technical University of Liberec,
Studentská 2, 461 17 Liberec, Czech Republic
Tel.: +420-48-535-2407
Fax: +420-48-535-2332
E-mail: dana.cerna@tul.cz

V. Finěk
Department of Mathematics and Didactics of Mathematics, Technical University of Liberec,
Studentská 2, 461 17 Liberec, Czech Republic

Biorthogonal spline-wavelet bases on the unit interval were constructed in [16]. The disadvantage of them is their bad condition which causes problems in practical applications. Some modifications which lead to better conditioned bases were proposed in [2], [17], [24], and [33]. The recent construction by M. Primbs, see [12], [24], or [25], seems to outperform the previous constructions with respect to the Riesz bounds as well as spectral properties of the corresponding stiffness matrices in the case of linear and quadratic spline-wavelets. In this paper, we focus on cubic spline wavelets and we construct interval spline-wavelet bases with the condition number which is close to the condition number of the spline wavelet bases on the real line. It is known that the condition number of the wavelet basis on the real line is less than or equal to the condition number of the interval wavelet basis, where the inner functions are restrictions of scaling functions and wavelets on the real line.

First of all, we summarize the desired properties:

– *Riesz basis property.* The functions form a Riesz basis of the space $L^2([0,1])$.
– *Locality.* The basis functions are local. Then the corresponding decomposition and reconstruction algorithms are simple and fast.
– *Biorthogonality.* The primal and dual wavelet bases form a biorthogonal pair.
– *Polymial exactness.* The primal MRA has polynomial exactness of order N and the dual MRA has polynomial exactness of order $\tilde{N}$. As in [9], $N+\tilde{N}$ has to be even and $\tilde{N} \geq N$.
– *Smoothness.* The smoothness of primal and dual wavelet bases is another desired property. It ensures the validity of norm equivalences, for details see below.
– *Closed form.* The primal scaling functions and wavelets are known in the closed form. It is a desirable property for the fast computation of integrals involving primal scaling functions and wavelets.
– *Well-conditioned bases.* Our objective is to construct wavelet bases with an improved condition number, especially for larger values of N and $\tilde{N}$.

From the viewpoint of numerical stability, ideal wavelet bases are orthogonal wavelet bases. However, they are usually avoided in the numerical treatment of partial differential and integral equations, because they are not accessible analytically, the complementary boundary conditions can not be satisfied and it is not possible to increase the number of vanishing wavelet moments independent from the order of accuracy. Moreover, sufficiently smooth orthogonal wavelets typically have a large support.

Biorthogonal wavelet bases on the unit interval derived from B-splines were constructed also in [8] and [19] and they were adapted to homogeneous Dirichlet boundary conditions in [20]. These bases are well-conditioned, but have globally supported dual basis functions. Another construction of spline-wavelets was proposed in [4], but the corresponding dual bases are unknown so far. We should also mention the construction of spline multiwavelets [15], [22], and [28], though the dual wavelets have a low Sobolev regularity.

The paper is organized as follows. Section 2 provides a short introduction to the concept of wavelet bases. Section 3 is concerned with the construction of primal multiresolution analysis on the interval. The primal scaling functions are B-splines defined on the Schoenberg sequence of knots, which have been used also in [4], [8], and [24]. In Section 4 we construct dual multiresolution analysis. There are two types of boundary scaling functions. The functions of the first type are defined in order to preserve the full degree of polynomial exactness as in [1] and [10]. The construction of the scaling functions of the second type is a delicate task, because the low condition number and nestedness of the multiresolution spaces have to be preserved. Section 5 is concerned with the computation of refinement matrices. In Section 6 wavelets are constructed by the method of stable completion proposed in [18]. The construction of initial stable completion is along the lines of [16]. In Section 7 we show that the constructed set of functions is indeed a Riesz basis for the space $L^2([0,1])$ and for the Sobolev space $H^s([0,1])$ for a certain range of s. In Section 8 we adapt the primal bases to the homogeneous Dirichlet boundary conditions of the first order and the dual bases to the complementary boundary conditions. Quantitative properties of the constructed bases are presented in Section 9. Finally, in Section 10, we compare the efficiency of an adaptive wavelet scheme for several spline-wavelet bases and we show a superiority of our construction. Numerical examples are presented for one-dimensional and two-dimensional Poisson equations where the solution has steep gradients.

2 Wavelet bases

This section provides a short introduction to the concept of wavelet bases. Let us introduce some notation. We use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ to denote the set of positive integers, integers, rational numbers, and real numbers, respectively. Let $\mathbb{N}_{j_0}$ denote the set of integers which are greater than or equal to j_0.

We consider a domain $\Omega \subset \mathbb{R}^d$ and the space $L^2(\Omega)$ with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Let $\mathscr{J}$ be some index set and let each index $\lambda \in \mathscr{J}$ takes the form $\lambda = (j,k)$, where $|\lambda| = j \in \mathbb{Z}$ is a *scale* or a *level*. Let $l^2(\mathscr{J})$ be a space of all sequences $b = \{b_\lambda\}_{\lambda \in \mathscr{J}}$ such that

$$\|b\|_{l_2(\mathscr{J})} := \left(\sum_{\lambda \in \mathscr{J}} |b_\lambda|^2 \right)^{\frac{1}{2}} < \infty. \tag{1}$$

Definition 1. A family $\Psi := \{\psi_\lambda \in \mathscr{J}\} \subset L^2(\Omega)$ is called a *wavelet basis* of $L^2(\Omega)$, if

i) Ψ is a *Riesz basis* for $L^2(\Omega)$, it means that the linear span of Ψ is dense in $L^2(\Omega)$ and there exist constants $c, C \in (0,\infty)$ such that

$$c\,\|\mathrm{b}\|_{l_2(\mathscr{J})} \le \left\| \sum_{\lambda \in \mathscr{J}} b_\lambda \psi_\lambda \right\| \le C\,\|\mathrm{b}\|_{l_2(\mathscr{J})} \quad \text{for all} \quad \mathrm{b} = \{b_\lambda\}_{\lambda \in \mathscr{J}} \in l^2(\mathscr{J}). \tag{2}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies } (2)\}, C_\psi := \inf\{C : C \text{ satisfies } (2)\}$ are called *Riesz bounds* and *cond* $\Psi = C_\psi / c_\psi$ is called the *condition number* of Ψ.

ii) The functions are *local* in the sense that

$$\operatorname{diam}(\Omega_\lambda) \le C 2^{-|\lambda|} \quad \text{for all} \quad \lambda \in \mathscr{J}, \tag{3}$$

where Ω_λ is the support of ψ_λ, and at a given level j the supports of only finitely many wavelets overlap in any point $x \in \Omega$.

By the Riesz representation theorem, there exists a unique family $\tilde{\Psi} = \{\tilde{\psi}_\lambda, \lambda \in \tilde{\mathscr{J}}\} \subset L^2(\Omega)$ biorthogonal to Ψ, i.e.

$$\langle \psi_{i,k}, \tilde{\psi}_{j,l} \rangle = \delta_{i,j} \delta_{k,l}, \quad \text{for all} \quad (i,k) \in \mathscr{J}, \quad (j,l) \in \tilde{\mathscr{J}}. \tag{4}$$

Here, $\delta_{i,j}$ denotes the Kronecker delta, i.e. $\delta_{i,i} := 1$, $\delta_{i,j} := 0$ for $i \neq j$. This family is also a Riesz basis for $L^2(\Omega)$. The basis Ψ is called a *primal* wavelet basis, $\tilde{\Psi}$ is called a *dual* wavelet basis.

In many cases, the wavelet system Ψ is constructed with the aid of a multiresolution analysis.

Definition 2. A sequence $S = \{S_j\}_{j \in \mathbb{N}_{j_0}}$ of closed linear subspaces $S_j \subset L^2(\Omega)$ is called a *multiresolution* or *multiscale analysis*, if

$$S_{j_0} \subset S_{j_0+1} \subset \ldots \subset S_j \subset S_{j+1} \subset \ldots L^2(\Omega) \quad \text{and} \quad \overline{\left(\cup_{j \in \mathbb{N}_{j_0}} S_j \right)} = L^2(\Omega). \tag{5}$$

The nestedness and the closedness of the multiresolution analysis implies the existence of the *complement spaces* W_j such that

$$S_{j+1} = S_j \oplus W_j, \tag{6}$$

where $\oplus$ denotes the direct sum.

We now assume that S_j and W_j are spanned by sets of basis functions

$$\Phi_j := \{\phi_{j,k}, k \in \mathscr{I}_j\}, \quad \Psi_j := \{\psi_{j,k}, k \in \mathscr{J}_j\}, \tag{7}$$

where $\mathscr{I}_j$, $\mathscr{J}_j$ are finite or at most countable index sets. We refer to $\phi_{j,k}$ as *scaling functions* and $\psi_{j,k}$ as *wavelets*. The multiscale basis is given by

$$\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j \tag{8}$$

and the overall wavelet basis of $L^2(\Omega)$ is obtained by

$$\Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j. \tag{9}$$

The single-scale and the multiscale bases are interrelated by the *wavelet transform* $\mathbf{T}_{j,s} : l^2(I_{j+s}) \to l^2(I_{j+s})$,

$$\Psi_{j,s} = \mathbf{T}_{j,s} \Phi_{j+s}. \tag{10}$$

The dual wavelet system $\tilde{\Psi}$ generates a dual multiresolution analysis $\tilde{S}$ with a dual scaling basis $\tilde{\Phi}$.

Polynomial exactness of order $N \in \mathbb{N}$ for the primal scaling basis and of order $\tilde{N} \in \mathbb{N}$ for the dual scaling basis is another desired property of wavelet bases. It means that

$$\Pi_{N-1}(\Omega) \subset S_j, \quad \Pi_{\tilde{N}-1}(\Omega) \subset \tilde{S}_j, \quad j \geq j_0, \tag{11}$$

where $\Pi_m(\Omega)$ is the space of all algebraic polynomials on Ω of a degree at most m.

3 Primal Scaling Basis

The primal scaling bases will be the same as bases designed by Chui and Quak in [8], because they are known to be well-conditioned. A big advantage of this approach is that it readily adapts to the bounded interval by introducing multiple knots at the endpoints. Let N be the desired order of the polynomial exactness of the primal scaling basis and let $\mathbf{t}^j = \left(t_k^j\right)_{k=-N+1}^{2^j+N-1}$ be a *Schoenberg sequence of knots* defined by

$$\begin{aligned} t_k^j &:= 0, \qquad k = -N+1, \ldots, 0, \\ t_k^j &:= \frac{k}{2^j}, \quad k = 1, \ldots, 2^j - 1, \\ t_k^j &:= 1, \qquad k = 2^j, \ldots, 2^j + N - 1. \end{aligned} \tag{12}$$

The corresponding *B-splines of order N* are defined by

$$B_{k,N}^j(x) := \left(t_{k+N}^j - t_k^j\right)\left[t_k^j, \ldots, t_{k+N}^j\right](t-x)_+^{N-1}, \quad x \in \langle 0, 1 \rangle, \tag{13}$$

where $(x)_+ := \max\{0, x\}$. The symbol $[t_k, \ldots t_{k+N}] f$ is the N-th divided difference of f which is recursively defined as

$$[t_k, \ldots, t_{k+N}] f = \begin{cases} \frac{[t_{k+1}, \ldots, t_{k+N}] f - [t_k, \ldots, t_{k+N-1}] f}{t_{k+N} - t_k} & \text{if} \quad t_k \neq t_{k+N}, \\ \frac{f^{(N)}(t_k)}{N!} & \text{if} \quad t_k = t_{k+N}, \end{cases} \tag{14}$$

with $[t_k] f = f(t_k)$.

The set $\Phi_j = \left\{\phi_{j,k}, k = -N+1, \ldots, 2^j - 1\right\}$ of primal scaling functions is then simply defined by

$$\phi_{j,k} = 2^{j/2} B_{k,N}^j, \quad k = -N+1, \ldots, 2^j - 1, \quad j \geq 0. \tag{15}$$

Thus there are $2^j - N + 1$ inner scaling functions and $N - 1$ functions at each boundary. Figure 1 shows the primal scaling functions for $N = 4$ and $j = 3$. The inner scaling functions are translations and dilations of one function ϕ which corresponds to the primal scaling function constructed by Cohen, Daubechies, and Feauveau in [9]. In the following, we consider ϕ from [9] which is shifted so that its support is $[0, N]$.

We define the primal multiresolution spaces by

$$S_j := \text{span}\, \Phi_j. \tag{16}$$

Lemma 3. *Under the above assumptions, the following holds:*

i) For any $j_0 \in \mathbb{N}$ the sequence $\mathscr{S} = \left\{S_j\right\}_{j \geq j_0}$ forms a multiresolution analysis of $L^2([0,1])$.

ii) The spaces S_j are exact of order N, i.e.

$$\Pi_{N-1}([0,1]) \subset S_j, \quad j \geq 1. \tag{17}$$

The proof can be found in [8], [24], [29].

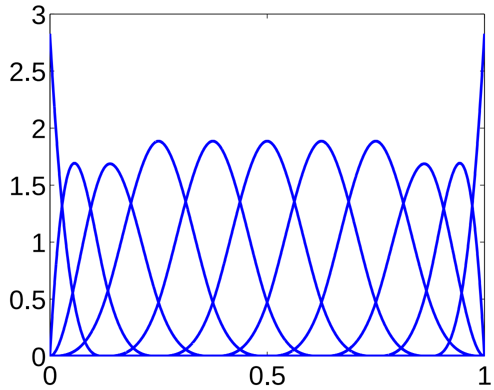

Fig. 1 Primal scaling functions for $N=4$ and $j=3$ without boundary conditions.

4 Dual Scaling Basis

The desired property of the dual scaling basis $\tilde{\Phi}$ is the biorthogonality to Φ and the polynomial exactness of order $\tilde{N}$. Let $\tilde{\phi}$ be the dual scaling function which was designed by Cohen, Daubechies, and Feauveau in [9] and which is shifted so that $\langle \phi, \tilde{\phi} \rangle = 0$, i.e. its support is $\left[-\tilde{N}+1, N+\tilde{N}-1\right]$. In this case $\tilde{N} \geq N$ and $\tilde{N}+N$ has to be an even number. It is known that there exist sequences $\{h_k\}_{k\in\mathbb{Z}}$ and $\{\tilde{h}_k\}_{k\in\mathbb{Z}}$ such that the functions ϕ and $\tilde{\phi}$ satisfy the *refinement equations*

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k \phi(2x-k), \quad \tilde{\phi}(x) = \sum_{k\in\mathbb{Z}} \tilde{h}_k \tilde{\phi}(2x-k), \quad x \in \mathbb{R}. \tag{18}$$

The parameters h_k and $\tilde{h}_k$ are called *scaling coefficients*. By biorthogonality of ϕ and $\tilde{\phi}$, we have

$$2 \sum_{k\in\mathbb{Z}} h_{2m+k} \tilde{h}_k = \delta_{0,m}, \quad m \in \mathbb{Z}. \tag{19}$$

Note that only coefficients $h_0, \ldots, h_N$ and $\tilde{h}_{-\tilde{N}+1}, \ldots, \tilde{h}_{N+\tilde{N}-1}$ may be nonzero.

In the sequel, we assume that

$$j \geq j_0 := \left\lceil \log_2 \left(N + 2\tilde{N} - 3\right) \right\rceil \tag{20}$$

so that the supports of the boundary functions are contained in $[0,1]$. We define inner scaling functions as translations and dilations of $\tilde{\phi}$:

$$\theta_{j,k} := 2^{j/2} \tilde{\phi}\left(2^j \cdot - k\right), \quad k = \tilde{N}-1, \ldots, 2^j - N - \tilde{N} + 1. \tag{21}$$

There will be two types of basis functions at each boundary. In the following, it will be convenient to abbreviate the boundary and inner index sets by

$$\mathscr{I}_j^{L,1} = \left\{-N+1, \ldots, -N+\tilde{N}\right\}, \tag{22}$$
$$\mathscr{I}_j^{L,2} = \left\{-N+\tilde{N}+1, \ldots, \tilde{N}-2\right\}, \tag{23}$$
$$\mathscr{I}_j^{0} = \left\{\tilde{N}-1, \ldots, 2^j - N - \tilde{N} + 1\right\}, \tag{24}$$
$$\mathscr{I}_j^{R,2} = \left\{2^j - N - \tilde{N} + 2, \ldots 2^j - \tilde{N} - 1\right\}, \tag{25}$$
$$\mathscr{I}_j^{R,1} = \left\{2^j - \tilde{N}, \ldots, 2^j - 1\right\}, \tag{26}$$

and

$$\mathscr{I}_j^{L} = \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{L,2} = \left\{-N+1, \ldots, \tilde{N}-2\right\}, \tag{27}$$
$$\mathscr{I}_j^{R} = \mathscr{I}_j^{R,2} \cup \mathscr{I}_j^{R,1} = \left\{2^j - N - \tilde{N} + 2, \ldots, 2^j - 1\right\}, \tag{28}$$
$$\mathscr{I}_j = \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{L,2} \cup \mathscr{I}_j^{0} \cup \mathscr{I}_j^{R,2} \cup \mathscr{I}_j^{R,1} = \left\{-N+1, \ldots, 2^j - 1\right\}. \tag{29}$$

Basis functions of the first type are defined to preserve polynomial exactness by the same way as in [1], [10]:

$$\theta_{j,k} = 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \langle p_{k+N-1}, \phi(\cdot - l)\rangle \tilde{\phi}\left(2^j \cdot -l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,1}, \tag{30}$$

where $\left\{p_0, \ldots, p_{\tilde{N}-1}\right\}$ is a basis of $\Pi_{\tilde{N}-1}([0,1])$. In Lemma 6 we show that the resulting dual scaling functions do not depend on the choice of the polynomial basis. In our case, p_k are the Bernstein polynomials defined by

$$p_k(x) := b^{-\tilde{N}+1} \binom{\tilde{N}-1}{k} x^k (b-x)^{\tilde{N}-1-k}, \quad k = 0, \ldots, \tilde{N}-1, \quad x \in \mathbb{R}. \tag{31}$$

The Bernstein polynomials were used also in [16]. On the contrary to [16], in our case the choice of polynomials does not affect the resulting dual scaling basis $\tilde{\Psi}$, but it has only the effect of stabilization of the computation, for details see Lemma 6 and the discussion below.

The definition of basis functions of the second type is a delicate task, because the low condition number and the nestedness of the multiresolution spaces have to be preserved. This means that the relation $\theta_{j,k} \subset \tilde{V}_j \subset \tilde{V}_{j+1}$, $k \in \mathscr{I}_j^{L,2}$, should hold. Therefore we define $\theta_{j,k}$, $k \in \mathscr{I}_j^{L,2}$, as linear combinations of functions which are already in $\tilde{V}_{j+1}$. To obtain well-conditioned bases, we define functions $\theta_{j,k}$, $k \in \mathscr{I}_j^{L,2}$, which are close to $\tilde{\phi}_{j,k}^{\mathbb{R}} := 2^{j/2}\tilde{\phi}\left(2^j \cdot -k\right)$, because $\tilde{\phi}_{j,k}^{\mathbb{R}}$, $k \in \mathscr{I}_j^{L,2}$, are biorthogonal to the inner primal scaling functions and the condition of $\left\{\tilde{\phi}_{j,k}^{\mathbb{R}}, k \in \mathscr{I}_j^{L,2} \cup \mathscr{I}_j^0\right\}$ is the same as the condition of the set of inner dual basis functions.

For this reason, we define the basis functions of the second type by

$$\theta_{j,k} = 2^{\frac{j}{2}} \sum_{l=\tilde{N}-1-2k}^{N+\tilde{N}-1} \tilde{h}_l \tilde{\phi}\left(2^{j+1} \cdot -2k - l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,2}, \tag{32}$$

where $\tilde{h}_i$ are the scaling coefficients corresponding to the scaling function $\tilde{\phi}$. Then $\theta_{j,k}$ is close to $\tilde{\phi}_{j,k}^{\mathbb{R}}|_{[0,1]}$, because by (18) we have

$$\tilde{\phi}_{j,k}^{\mathbb{R}}|_{[0,1]} = 2^{\frac{j}{2}} \sum_{k=-\tilde{N}+1}^{N+\tilde{N}-1} \tilde{h}_l \tilde{\phi}\left(2^{j+1} \cdot -2k - l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,2}. \tag{33}$$

Figure 2 shows the functions $\theta_{j,k}$ and $\tilde{\phi}_{j,k}^{\mathbb{R}}$ for $N = 4$, $\tilde{N} = 6$, and $j = 4$.

The boundary functions at the right boundary are defined to be symmetric with the left boundary functions:

$$\theta_{j,k} = \theta_{j,2^j-k}(1-\cdot), \quad k \in \mathscr{I}_j^R. \tag{34}$$

It is easy to see that

$$\theta_{j+1,k} = 2^{1/2}\theta_{j,k}(2\cdot), \quad k \in \mathscr{I}_j^L \tag{35}$$

for the left boundary functions and

$$\theta_{j+1,k}(1-\cdot) = 2^{1/2}\theta_{j,k}(1-2\cdot), \quad k \in \mathscr{I}_j^R \tag{36}$$

for the right boundary functions.

Since the set $\Theta_j := \left\{\theta_{j,k}, k \in \mathscr{I}_j\right\}$ is not biorthogonal to Φ_j, we derive a new set

$$\tilde{\Phi}_j := \left\{\tilde{\phi}_{j,k}, k \in \mathscr{I}_j\right\} \tag{37}$$

from Θ_j by biorthogonalization. Let

$$\mathbf{Q}_j = \left(\langle \phi_{j,k}, \theta_{j,l}\rangle\right)_{k,l \in \mathscr{I}_j}. \tag{38}$$

Then viewing $\tilde{\Phi}_j$ and Θ_j as column vectors we define

$$\tilde{\Phi}_j := \mathbf{Q}_j^{-T}\Theta_j, \tag{39}$$

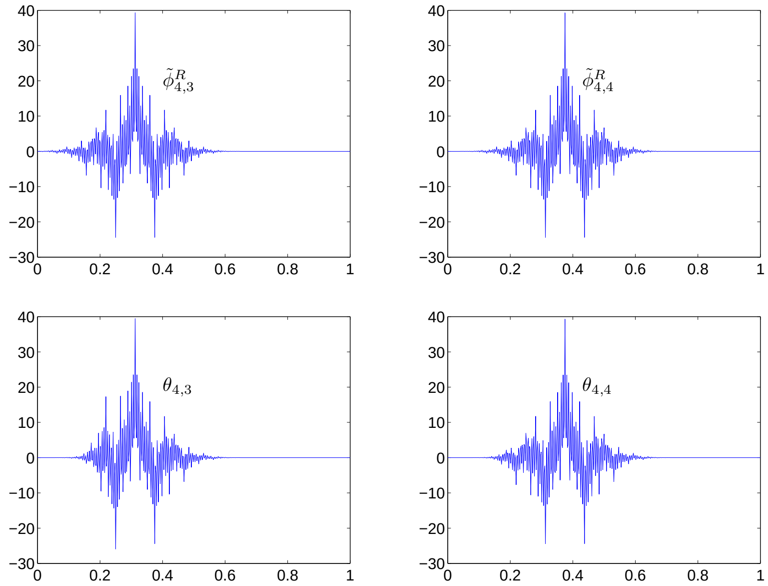

Fig. 2 The functions $\tilde{\phi}^{\mathbb{R}}_{4,k}$ and $\theta_{4,k}$ for $N = 4$ and $\tilde{N} = 6$.

assuming that $\mathbf{Q}_j$ is invertible, which is the case of all choices of N and $\tilde{N}$ considered in our numerical examples below.

Then $\tilde{\Phi}_j$ is biorthogonal to Φ_j, because

$$\left\langle \Phi_j, \tilde{\Phi}_j \right\rangle = \left\langle \Phi_j, \mathbf{Q}_j^{-T} \Theta_j \right\rangle = \mathbf{Q}_j \mathbf{Q}_j^{-1} = \mathbf{I}_{\#\mathscr{I}_j}, \tag{40}$$

where the symbol # denotes the cardinality of the set and $\mathbf{I}_m$ denotes the identity matrix of the size $m \times m$.

Lemma 4. *i) Let Φ_j, Θ_j be defined as above. Then the matrices*

$$\mathbf{Q}_{j,L} = \left(\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle \right)_{k,l \in \mathscr{I}_j^L} \quad \text{and} \quad \mathbf{Q}_{j,R} = \left(\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle \right)_{k,l \in \mathscr{I}_j^R} \tag{41}$$

are independent of j, i.e. there are matrices $\mathbf{Q}_L$, $\mathbf{Q}_R$ such that

$$\mathbf{Q}_{j,L} = \mathbf{Q}_L, \quad \mathbf{Q}_{j,R} = \mathbf{Q}_R. \tag{42}$$

Moreover, the matrix $\mathbf{Q}_R$ results from the matrix $\mathbf{Q}_L$ by reversing the ordering of rows and columns, which means that

$$(\mathbf{Q}_R)_{k,l} = (\mathbf{Q}_L)_{2^j-N-k, 2^j-N-l}, \quad k,l \in \mathscr{I}_j^R. \tag{43}$$

ii) The following holds:

$$\left(\mathbf{Q}_j\right)_{k,l} = \delta_{k,l}, \quad k \in \mathscr{I}_j, \ l \in \mathscr{I}_j^0. \tag{44}$$

iii) The following holds:

$$\left(\mathbf{Q}_j\right)_{k,l} = 0, \quad k \in \mathscr{I}_j^0, \ l \in \mathscr{I}_j^L \cup \mathscr{I}_j^R. \tag{45}$$

Proof Due to (35) and by substitution we have for $k,l \in \mathscr{I}_j^L$

$$\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle = \left\langle 2^{\frac{j-j_0}{2}} \phi_{j_0,k} \left(2^{j-j_0} \cdot \right), 2^{\frac{j-j_0}{2}} \theta_{j_0,l} \left(2^{j-j_0} \cdot \right) \right\rangle = \left\langle \phi_{j_0,k}, \theta_{j_0,l} \right\rangle. \tag{46}$$

Therefore, $\mathbf{Q}_{j,L} = \mathbf{Q}_{j_0,L} = \mathbf{Q}_L$, i.e. the matrix $\mathbf{Q}_{j,L}$ is independent of j. Due to (36) $\mathbf{Q}_{j,R}$ is independent of j too. The property (43) is a direct consequence of the reflection invariance (34).

The property *ii*) follows from the biorthogonality of $\{\phi(\cdot-k)\}_{k\in\mathbb{Z}}$ and $\left\{\tilde{\phi}(\cdot-l)\right\}_{l\in\mathbb{Z}}$. It also implies (45) for $k\in\mathscr{I}_j^0$, $l\in\mathscr{I}_j^{L,1}\cup\mathscr{I}_j^{R,1}$. It remains to prove (45) for $k\in\mathscr{I}_j^0$, $l\in\mathscr{I}_j^{L,2}\cup\mathscr{I}_j^{R,2}$. By the definition of the dual scaling functions of the second type (32), the refinement relation (18) for the dual scaling function $\tilde{\phi}$, and (19), we have for $k\in\mathscr{I}_j^0$, $l\in\mathscr{I}_j^{L,2}$,

$$\langle \phi_{j,k},\theta_{j,l}\rangle = \left\langle \phi(\cdot-k), \sqrt{2}\sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1}\tilde{h}_l\tilde{\phi}(2\cdot-2l-m)|_{[0,1]}\right\rangle \tag{47}$$

$$= 2\left\langle \sum_{n=0}^{N} h_n\phi(2\cdot-2k-n), \sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1}\tilde{h}_m\tilde{\phi}(2\cdot-2l-m)|_{[0,1]}\right\rangle \tag{48}$$

$$= 2\sum_{n=0}^{N}\sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} h_n\tilde{h}_m\delta_{2k+n,2l+m} = 2\sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} h_{2l-2k+m}\tilde{h}_m \tag{49}$$

$$= 2\sum_{m\in\mathbb{Z}} h_{2l-2k+m}\tilde{h}_m = 0. \tag{50}$$

By (34), the relation (45) holds also for $k\in\mathscr{I}_j^0$, $l\in\mathscr{I}_j^{R,2}$.

Thus, we can write

$$\tilde{\Phi}_j := \mathbf{Q}_j^{-T}\Theta_j = \begin{pmatrix}\mathbf{Q}_L & & \\ & \mathbf{I}_{\#\mathscr{I}_j^0} & \\ & & \mathbf{Q}_R\end{pmatrix}^{-T}\Theta_j = \begin{pmatrix}\mathbf{Q}_L^{-T} & & \\ & \mathbf{I}_{\#\mathscr{I}_j^0} & \\ & & \mathbf{Q}_R^{-T}\end{pmatrix}\Theta_j, \tag{51}$$

Since the matrix $\mathbf{Q}_j$ is symmetric in the sense of (43), the properties (34), (35), and (36) hold for $\tilde{\phi}_{j,k}$ as well.

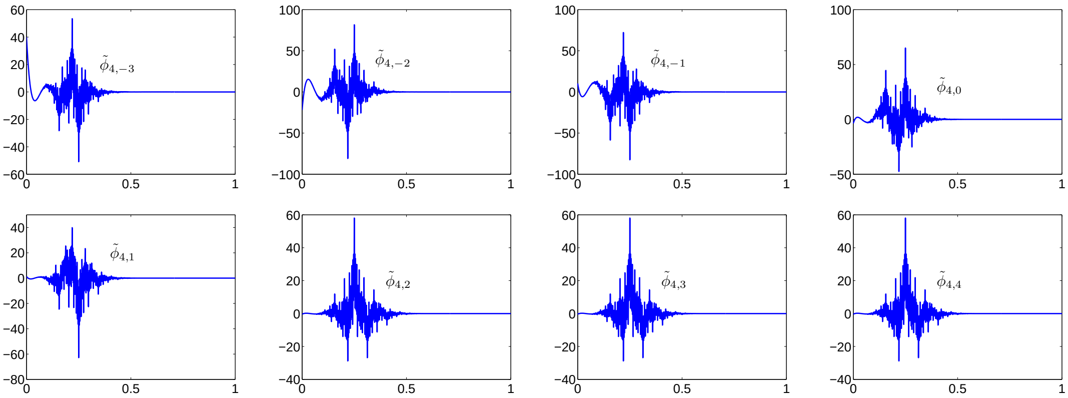

Fig. 3 Boundary dual scaling functions for $N=4$ and $\tilde{N}=6$ without boundary conditions.

Remark 5. *It is known that the scaling function $\tilde{\phi}$ has typically a low Sobolev regularity for smaller values of $\tilde{N}$. Thus the functions $\theta_{j,k}$ have a low Sobolev regularity for smaller values of $\tilde{N}$, too. Therefore, we do not obtain the sufficiently accurate entries of the matrix $\mathbf{Q}_j$ directly by classical quadratures. Fortunately, we are able to compute the matrix $\mathbf{Q}_j$ precisely up to the round off errors. For $k\in\mathscr{I}_j^{L,1}\cup\mathscr{I}_j^{L,2}$, $l\in\mathscr{I}_j^{L,1}$ we have*

$$\langle \phi_{j,k},\theta_{j,l}\rangle = \sum_{m=-N-\tilde{N}+2}^{\tilde{N}-2}\sum_{n=0}^{\tilde{N}-1} c_{l,n}\langle(\cdot)^n,\phi(\cdot-m)\rangle\left\langle\phi(\cdot-k),\tilde{\phi}(\cdot-m)\right\rangle_{L^2(\langle 0,1\rangle)}, \tag{52}$$

with $c_{l,n}$ given by (64)*. Since ϕ is a piecewise polynomial function and $\tilde{\phi}$ is refinable, for $k \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{L,2}$, $l \in \mathscr{I}_j^{L,1}$ we can compute the entries of $\mathbf{Q}_j$ by the method from [11]. By the refinement relation we easily obtain the following relations for the computation of the remaining entries of $\mathbf{Q}^L$:*

$$\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle = \begin{cases} \sum_{m=\tilde{N}-1-2l}^{N+\tilde{N}-1} \tilde{h}_m \left\langle \phi_{0,k}, \tilde{\phi}\left(\cdot - 2k - m\right) \right\rangle, & k = -N+1, \ldots, -1,\ l \in \mathscr{I}_j^{L,2}, \\ 2^{-1} \sum_{m=\tilde{N}-1-2l}^{N+\tilde{N}-1} h_{2k-2l+m} \tilde{h}_m, & k = 0, \ldots, \tilde{N}-2,\ l \in \mathscr{I}_j^{L,2}. \end{cases} \tag{53}$$

Since the submatrix $\mathbf{Q}_R$ is obtained from a matrix $\mathbf{Q}_L$ by reversing the ordering of rows and columns, the matrix $\mathbf{Q}_j$ can be indeed computed precisely up to the round off errors.

Now we show that the resulting dual scaling basis $\tilde{\Phi}$ does not depend on a choice of a polynomial basis of the space $\Pi_{\tilde{N}}\left([0,1]\right)$ in the formula (30).

Lemma 6. *We suppose that $P^1 = \left\{p_0^1, \ldots, p_{\tilde{N}-1}^1\right\}$, $P^2 = \left\{p_0^2, \ldots, p_{\tilde{N}-1}^2\right\}$ are two different bases of the space $\Pi_{\tilde{N}}\left([0,1]\right)$ and for $i = 1,2$ we define the sets $\Theta_j^i = \left\{\theta_{j,k}^i\right\}_{k=-N+1}^{2^j-1}$ by*

$$\theta_{j,k}^i = \begin{cases} 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \left\langle p_{k+N-1}^i, \phi\left(\cdot - l\right)\right\rangle \tilde{\phi}\left(2^j \cdot - l\right)|_{[0,1]}, & k \in \mathscr{I}_j^{L,1}, \\ \theta_{j,2^j-N-k}^i, & k \in \mathscr{I}_j^{R,1}, \\ \theta_{j,k}, & k \in \mathscr{I}_j^{L,2} \cup \mathscr{I}_j^{0} \cup \mathscr{I}_j^{R,2}. \end{cases} \tag{54}$$

Furthermore, we define

$$\mathbf{Q}_j^i = \left\langle \Phi_j, \Theta_j^i \right\rangle, \quad \tilde{\Phi}_j^i = \left(\mathbf{Q}_j^i\right)^{-T} \Theta_j^i, \quad i = 1,2, \tag{55}$$

and we assume that $\mathbf{Q}_j^i$ is nonsingular. Then $\tilde{\Phi}_j^1 = \tilde{\Phi}_j^2$.

Proof Since P^1 and P^2 are both bases of the space $\Pi_{\tilde{N}}\left([0,1]\right)$, there exists a regular matrix $\mathbf{B}_L$ such that $P^2 = \mathbf{B}_L P^1$. The consequence is that

$$\Theta^2 = \mathbf{B}_j \Theta^1, \tag{56}$$

with

$$\mathbf{B}_j = \begin{pmatrix} \mathbf{B}_L & & \\ & \mathbf{I}_{\#\mathscr{I}_j^0} & \\ & & \mathbf{B}_R \end{pmatrix}, \tag{57}$$

where $\mathbf{B}_R$ results from a matrix $\mathbf{B}_L$ by reversing the ordering of rows and columns, which means that

$$\left(\mathbf{B}_R\right)_{k,l} = \left(\mathbf{B}_L\right)_{2^j-N-k,2^j-N-l}, \quad k,l \in \mathscr{I}_j^{L,1}. \tag{58}$$

Therefore, we have

$$\tilde{\Phi}_j^2 = \left(\mathbf{Q}_j^2\right)^{-T} \Theta_j^2 = \left(\mathbf{Q}_j^1\right)^{-T} \mathbf{B}_j^{-1} \mathbf{B}_j \Theta_j^1 = \tilde{\Phi}_j^1. \tag{59}$$

Although a choice of a polynomial basis does not influence the resulting dual scaling basis, it has an influence on the stability of the computation and the preciseness of the results, because some choices of the polynomial bases lead to the critical values of the condition number of the biorthogonalization matrix. We present the condition numbers of the matrix $\mathbf{Q}_L$ for the monomial basis $\left\{1, x, x^2, \ldots x^{\tilde{N}-1}\right\}$ and Bernstein polynomials (31) with the parameters $b = 4$ and $b = 10$ in Table 4. In our numerical experiments in Section 9 we choose $b = 10$.

Remark 7. *In the case of linear primal basis, i.e. $N = 2$, there are no boundary dual functions of the second type. In [24] the primal scaling functions and the inner dual scaling functions are the same as ours. The boundary dual functions before biorthogonalization are defined by (30) with the same choice of polynomials $p_0, \ldots, p_{\tilde{N}-1}$ as in [10]. Due to the Lemma 6, for $N = 2$ the wavelet basis in [24] is identical to the wavelet basis constructed in this section.*

The main difference of the construction by M. Primbs [24] in comparison with our construction is the definition of dual basis functions of the second type $\theta_{j,k}^{Primbs}$, $k = -N+1, \ldots, -2$. Note that they correspond to different indexes than ours. These functions are defined as linear combination of functions $\theta_{j+1,n}^{Primbs}$, $n > k$, in order to be *already biorthogonal to the primal scaling functions. The refinement coefficients for them are obtained by solving certain system of linear algebraic equations. In case $N > 3$, the functions $\theta_{j,k}^{Primbs}$, $k = -N+1,\ldots,-2$, take much larger values than primal scaling functions and than the inner dual scaling functions. Then some of the boundary wavelets take much larger values than inner wavelets which probably causes bad conditioning of wavelet bases. Furthermore, the dual boundary functions of the first type which are defined to preserve the polynomial exactness correspond to the first N scaling functions in our case and they correspond to the primal scaling functions indexed by $-1,\ldots,\tilde{N}-2$ in case of the construction from [24]. It leads to better matching of the supports and values of the primal and dual functions in our construction. This better localization and 'almost biorthogonality' of the dual functions of the second type to the primal scaling functions lead to optimally conditioned wavelet bases for $N \leq 4$ and to an improvement of the condition number also for $N = 5$, see Section 9.*

The constructions of primal and dual boundary scaling functions in [16] and [17] is based on the relation (30) with various choices of polynomials. There are no boundary generators of the second type. This construction also leads to some boundary functions which take larger values than the inner functions and the condition number of wavelet bases is bad for $N > 3$, see figures in [16], [17], and [35].

For the proof of Theorem 9 below and also for deriving of refinement matrices we will need the following lemma.

Lemma 8. *For the left boundary functions of the first type there exist refinement coefficients $m_{n,k}$, $k \in \mathscr{I}_j^{L,1}$, $n \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^3$, $\mathscr{I}_j^3 := \left\{\tilde{N}-1,\ldots,3\tilde{N}+N-5\right\}$ such that*

$$\theta_{j,k} = \sum_{n=-N+1}^{-N+\tilde{N}} m_{n,k}\theta_{j+1,n} + \sum_{n=\tilde{N}-1}^{3\tilde{N}+N-5} m_{n,k}\theta_{j+1,n}, \quad k \in \mathscr{I}_j^{L,1}. \tag{60}$$

Proof Let $\Theta_j^0 = \left\{\theta_{j,k}, k \in \mathscr{I}_j^3\right\}$ and $\Theta_j^{L,1,mon} = \left\{\theta_{j,k}^{mon}, k \in \mathscr{I}_j^{L,1}\right\}$ be defined by

$$\theta_{j,k}^{mon} = 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \left\langle (\cdot)^i, \phi(\cdot - l)\right\rangle \tilde{\phi}\left(2^j \cdot - l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,1}. \tag{61}$$

Then

$$\Theta_j^{L,1,mon} = \left(\mathbf{M}^{mon}\right)^T \begin{pmatrix} \Theta_{j+1}^{L,1,mon} \\ \Theta_{j+1}^0 \end{pmatrix}, \tag{62}$$

where the refinement matrix $\mathbf{M}^{mon} = \left\{m_{n,k}^{mon}\right\}_{n\in\mathscr{I}_j^{L,1}\cup\mathscr{I}_j^3, k\in\mathscr{I}_j^{L,1}}$ is given by

$$m_{n,k}^{mon} = \begin{cases} \frac{1}{\sqrt{2}} 2^{-k}, & k = n,\ n \in \mathscr{I}_j^{L,1}, \\ \frac{1}{\sqrt{2}} \sum_{q=\left\lceil \frac{n-N-\tilde{N}+1}{2} \right\rceil}^{\tilde{N}-2} \left\langle (\cdot)^{k+N-1}, \phi(\cdot - q)\right\rangle \tilde{h}_{n-2q}, & k \in \mathscr{I}_j^{L,1},\ n \in \mathscr{I}_j^3, \\ 0, & \text{otherwise.} \end{cases} \tag{63}$$

Table 1 Condition numbers of the matrices $\mathbf{Q}_L$

N	$\tilde{N}$	mon.	$b = 4$	$b = 10$	N	$\tilde{N}$	mon.	$b = 4$	$b = 10$
2	2	6.68e+00	9.94e+00	3.16e+01	4	4	2.46e+04	6.75e+02	1.33e+04
2	4	4.66e+02	1.94e+01	9.48e+02	4	6	1.30e+07	2.94e+04	7.34e+04
2	6	1.40e+05	1.00e+02	4.47e+03	4	8	1.24e+10	6.24e+06	9.42e+04
2	8	1.03e+08	8.52e+03	5.81e+03	4	10	1.92e+13	2.26e+09	5.24e+04
2	10	1.48e+11	1.67e+06	1.58e+03	5	5	5.34e+06	3.29e+04	1.26e+05
3	3	2.18e+02	1.07e+02	1.00e+03	5	7	5.62e+09	6.91e+06	3.73e+05
3	5	3.73e+04	1.88e+02	1.05e+04	5	9	9.39e+12	2.57e+09	3.47e+05
3	7	1.64e+07	1.20e+04	2.26e+04	6	6	1.20e+09	3.68e+06	6.81e+05
3	9	1.54e+10	2.90e+06	1.33e+04	6	8	2.97e+12	1.92e+09	1.81e+06

For deriving of $\mathbf{M}^{mon}$ see [16]. It is known that the coefficients of Bernstein polynomials in a monomial basis are given by

$$c_{l,n} = \begin{cases} (-1)^{l-n} \binom{\tilde{N}-1}{n} \binom{n}{l} b^{-n}, & \text{if} \quad n \geq l, \\ 0, & \text{otherwise.} \end{cases} \tag{64}$$

Hence, the matrix $\mathbf{C} = \left\{c_{l,n}\right\}_{l,n=-N+1}^{-N+\tilde{N}}$ is an upper triangular matrix with nonzero entries on the diagonal which implies that $\mathbf{C}$ is invertible. We denote $\Theta_j^{L,1} = \left\{\theta_{j,k}, k \in \mathscr{I}_j^{L,1}\right\}$ and we obtain

$$\Theta_j^{L,1} = \mathbf{C}\Theta_j^{L,1,mon} = \mathbf{C}\left(\mathbf{M}^{mon}\right)^T \begin{pmatrix} \Theta_{j+1}^{L,1,mon} \\ \Theta_{j+1}^{0} \end{pmatrix} = \mathbf{C}\left(\mathbf{M}^{mon}\right)^T \begin{pmatrix} \mathbf{C}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \Theta_{j+1}^{L,1} \\ \Theta_{j+1}^{0} \end{pmatrix}. \tag{65}$$

Therefore, the refinement matrix $\mathbf{M} = \left\{m_{n,k}\right\}_{n\in\mathscr{I}_j^{L,1}\cup\mathscr{I}_j^{3}, k\in\mathscr{I}_j^{L,1}}$ is given by

$$\mathbf{M} = \begin{pmatrix} \mathbf{C}^{-T} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \mathbf{M}^{mon}\mathbf{C}^{T}. \tag{66}$$

We define the dual multiresolution spaces by

$$\tilde{S}_j := \text{span } \tilde{\Phi}_j. \tag{67}$$

Theorem 9. *Under the above assumptions, the following holds*
i) The sequence $\tilde{\mathscr{S}} = \left\{\tilde{S}_j\right\}_{j\geq j_0}$ *forms a multiresolution analysis of* $L^2\left([0,1]\right)$.
ii) The spaces $\tilde{S}_j$ *are exact of order* $\tilde{N}$*, i.e.*

$$\Pi_{\tilde{N}-1}\left([0,1]\right) \subset \tilde{S}_j, \quad j > j_0. \tag{68}$$

Proof To prove $i)$ we have to show the nestedness of the spaces $\tilde{S}_j$, i.e. $\tilde{S}_j \subset \tilde{S}_{j+1}$. Note that

$$\tilde{S}_j = \text{span } \tilde{\Phi}_j = \text{span } \Theta_j. \tag{69}$$

Therefore, it is sufficient to prove that any function from Θ_j can be written as a linear combination of the functions from Θ_{j+1}. For the left boundary functions of the first type it is a consequence of Lemma 8. By definition (32) it holds also for the left boundary functions of the second type. Since the inner basis functions are just translated and dilated scaling function $\tilde{\phi}$, they obviously satisfy the refinement relation. Finally, the right boundary scaling functions are derived by the reflection from the left boundary scaling functions and therefore, they satisfy the refinement relation, too. It remains to prove that

$$\overline{\bigcup_{j\geq j_0} \tilde{S}_j} = L^2\left([0,1]\right), \tag{70}$$

where $\overline{M}$ denotes the closure of the set M in $L^2\left([0,1]\right)$. It is known [26] that for the spaces generated by inner functions

$$\tilde{S}_j^0 := \left\{\theta_{j,k}, k \in \mathscr{I}_j^0\right\} \tag{71}$$

we have

$$\overline{\bigcup_{j\geq j_0} \tilde{S}_j^0} = L^2\left([0,1]\right). \tag{72}$$

Hence, (70) holds independently of the choice of boundary functions.

To prove ii) we recall that the scaling function $\tilde{\phi}$ is exact of order $\tilde{N}$, i.e.

$$2^{j(r+1/2)}x^r = \sum_{k\in\mathbb{Z}} \alpha_{k,r} 2^{j/2}\tilde{\phi}\left(2^j x - k\right), \quad x \in \mathbb{R} \; a.e., \; r = 0, \ldots, \tilde{N}-1, \tag{73}$$

where

$$\alpha_{k,r} = \left\langle (\cdot)^k, \phi\left(\cdot - r\right)\right\rangle. \tag{74}$$

It implies that for $r = 0,\dots,\tilde{N}-1$, $x \in \langle 0,1\rangle$, the following holds

$$2^{j(r+1/2)}x^r|_{\langle 0,1\rangle} = \sum_{k=-N-\tilde{N}+2}^{\tilde{N}-2} \alpha_{k,r}2^{j/2}\tilde{\phi}\left(2^jx-k\right)|_{\langle 0,1\rangle} + \sum_{k=\tilde{N}-1}^{2^j-N-\tilde{N}+1} \alpha_{k,r}2^{j/2}\tilde{\phi}\left(2^jx-k\right)|_{\langle 0,1\rangle} + \sum_{k=2^j-N-\tilde{N}+2}^{2^j+\tilde{N}-2} \alpha_{k,r}2^{j/2}\tilde{\phi}\left(2^jx-k\right)|_{\langle 0,1\rangle}.$$

By (30), (34), and (69), we immediately have

$$\Pi_{\tilde{N}-1}([0,1]) \subset \operatorname{span}\left\{\tilde{\phi}_{j,k}, k \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^0 \cup \mathscr{I}_j^{R,1}\right\} \subset \tilde{S}_j. \tag{75}$$

5 Refinement Matrices

Due to the length of the support of the primal scaling functions, the refinement matrix $M_{j,0}$ corresponding to Φ has the following structure:

$$\mathbf{M}_{j,0} = \begin{pmatrix} \mathbf{M}_L & & \\ & \mathbf{A}_j & \\ & & \mathbf{M}_R \end{pmatrix}. \tag{76}$$

where $\mathbf{M}_L$, $\mathbf{M}_R$ are blocks of the size $(2N-2)\times(N-1)$ and $\mathbf{A}_j$ is a $\left(2^{j+1}-N+2\right)\times\left(2^j-N+2\right)$ matrix given by

$$(\mathbf{A}_j)_{m,n} = \frac{1}{\sqrt{2}}h_{m-2n}, \quad 0 \le m-2n \le N. \tag{77}$$

Since the matrix $\mathbf{M}_L$ is given by

$$\begin{pmatrix} \phi_{j,-N+1} \\ \phi_{j,-N+2} \\ \vdots \\ \phi_{j,-1} \end{pmatrix} = \mathbf{M}_L^T \begin{pmatrix} \phi_{j+1,-N+1} \\ \phi_{j+1,-N+2} \\ \vdots \\ \phi_{j+1,N-1} \end{pmatrix}, \tag{78}$$

it could be computed by solving the system

$$\mathbf{P}_1 = \mathbf{M}_L^T\mathbf{P}_2, \tag{79}$$

where

$$\mathbf{P}_1 = \begin{pmatrix} \phi_{0,-N+1}(0) & \phi_{0,-N+1}(1) & \dots & \phi_{0,-N+1}(2N-3) \\ \phi_{0,-N+2}(0) & \phi_{0,-N+2}(1) & \dots & \phi_{0,-N+2}(2N-3) \\ \vdots & & & \vdots \\ \phi_{0,-1}(0) & \phi_{0,-1}(1) & \dots & \phi_{0,-1}(2N-3) \end{pmatrix} \tag{80}$$

and

$$\mathbf{P}_2 = \begin{pmatrix} \phi_{1,-N+1}(0) & \phi_{1,-N+1}(1) & \dots & \phi_{1,-N+1}(2N-3) \\ \phi_{1,-N+2}(0) & \phi_{1,-N+2}(1) & \dots & \phi_{1,-N+2}(2N-3) \\ \vdots & & & \vdots \\ \phi_{1,N-1}(0) & \phi_{1,N-1}(1) & \dots & \phi_{1,N-1}(2N-3) \end{pmatrix}. \tag{81}$$

The solution of the system (79) exists and is unique if and only if the matrix $\mathbf{P}_2$ is nonsingular. The proof of a nonsingularity of $\mathbf{P}_2$ can be found in [36].

To compute the refinement matrix corresponding to the dual scaling functions, we need to identify first the structure of the refinement matrices $\mathbf{M}_{j,0}^{\Theta}$ corresponding to Θ.

$$\mathbf{M}_{j,0}^{\Theta} = \begin{pmatrix} \mathbf{M}_L^{\Theta} & & \\ & \tilde{\mathbf{A}}_j & \\ & & \mathbf{M}_R^{\Theta} \end{pmatrix}, \tag{82}$$

where $\mathbf{M}_L^{\Theta}$ and $\mathbf{M}_R^{\Theta}$ are blocks of the size $\left(2N+3\tilde{N}-5\right)\times\left(N+\tilde{N}-2\right)$ and $\tilde{\mathbf{A}}_j$ is a matrix of the size $\left(2^{j+1}-N-2\tilde{N}+3\right)\times\left(2^j-N-2\tilde{N}+3\right)$ given by

$$\left(\tilde{\mathbf{A}}_j\right)_{m,n} = \frac{1}{\sqrt{2}}\tilde{h}_{m-2n}, \quad 0 \le m-2n \le N+\tilde{N}-2. \tag{83}$$

The recipe for the computation of the refinement coefficients for the left boundary functions of the first type is the proof of Lemma 8. The refinement coefficients for the left boundary functions of the second type are given by the definition (32). The matrix $\mathbf{M}_R^{\Theta}$ can be computed by the similar way.

Since we have

$$\tilde{\Phi}_j = \mathbf{Q}_j^{-T}\Theta_j = \mathbf{Q}_j^{-T}\left(\mathbf{M}_{j,0}^{\Theta}\right)^T\Theta_{j+1} = \mathbf{Q}_j^{-T}\left(\mathbf{M}_{j,0}^{\Theta}\right)^T\mathbf{Q}_{j+1}^T\tilde{\Phi}_{j+1}, \tag{84}$$

the refinement matrix $\tilde{\mathbf{M}}_{j,0}$ corresponding to the dual scaling basis $\tilde{\Phi}_j$ is given by

$$\tilde{\mathbf{M}}_{j,0} = \mathbf{Q}_{j+1}\mathbf{M}_{j,0}^{\Theta}\mathbf{Q}_j^{-1}. \tag{85}$$

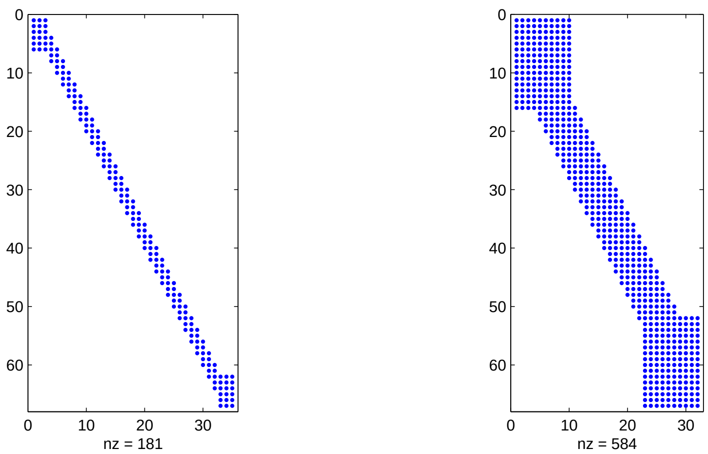

Fig. 4 Nonzero pattern of the matrices $\mathbf{M}_{5,0}$ and $\tilde{\mathbf{M}}_{5,0}$ for $N=4$ and $\tilde{N}=6$, *nz* is the number of nonzero entries.

6 Wavelets

Our next goal is to determine the corresponding wavelet bases. This is directly connected to the task of determining an appropriate matrices $\mathbf{M}_{j,1}$ and $\tilde{\mathbf{M}}_{j,1}$. Thus, the problem has been transferred from functional analysis to linear algebra. We follow a general principle called a *stable completion* which was proposed in [6].

Definition 10. Any $\mathbf{M}_{j,1} : l_2(J_j) \to l_2(I_{j+1})$ is called a *stable completion* of $\mathbf{M}_{j,0}$, if

$$\left\|\mathbf{M}_j\right\|, \left\|\mathbf{M}_j^{-1}\right\| = O(1), \quad j \to \infty, \tag{86}$$

where $\mathbf{M}_j := \left(\mathbf{M}_{j,0}, \mathbf{M}_{j,1}\right)$.

The idea is to determine first an initial stable completion and then to project it to the desired complement space W_j determined by $\left\{\tilde{V}_j\right\}_{j \geq j_0}$. This is summarized in the following theorem [6].

Theorem 11. *Let Φ_j and $\tilde{\Phi}_j$ be a primal and dual scaling basis, respectively. Let $\mathbf{M}_{j,0}$ and $\tilde{\mathbf{M}}_{j,0}$ be the refinement matrices corresponding to these bases. Suppose that $\check{\mathbf{M}}_{j,1}$ is some stable completion of $\mathbf{M}_{j,0}$ and $\check{\mathbf{G}}_j = \check{\mathbf{M}}_j^{-1}$. Then*

$$\mathbf{M}_{j,1} := \left(\mathbf{I} - \mathbf{M}_{j,0}\tilde{\mathbf{M}}_{j,0}^T\right)\check{\mathbf{M}}_{j,1} \tag{87}$$

is also a stable completion and $\mathbf{G}_j = \mathbf{M}_j^{-1}$ has the form

$$\mathbf{G}_j = \begin{pmatrix} \tilde{\mathbf{M}}_{j,0}^T \\ \check{\mathbf{G}}_{j,1} \end{pmatrix}. \tag{88}$$

Moreover, the collections

$$\Psi_j := \mathbf{M}_{j,1}^T \Phi_{j+1}, \quad \tilde{\Psi}_j := \check{\mathbf{G}}_{j,1}^T \tilde{\Phi}_{j+1} \tag{89}$$

form biorthogonal systems

$$\left\langle \Psi_j, \tilde{\Psi}_j \right\rangle = \mathbf{I}, \quad \left\langle \Phi_j, \tilde{\Psi}_j \right\rangle = \left\langle \Psi_j, \tilde{\Phi}_j \right\rangle = \mathbf{0}. \tag{90}$$

We found the initial stable completion by the method from [16], [18] with some small changes. The difference is only in the dimensions of the involved matrices and in the definion of the matrix $\mathbf{F}_j$. Recall that $\mathbf{A}_j$ is the interior block in the matrix $\mathbf{M}_{j,0}$ of the form

$$\mathbf{A}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} h_0 & 0 & \dots & 0 \\ h_1 & 0 & & \vdots \\ h_3 & h_0 & & \\ \vdots & \vdots & & \vdots \\ h_N & h_{N-2} & & \vdots \\ 0 & h_{N-1} & & 0 \\ 0 & h_N & & h_0 \\ \vdots & & & \vdots \\ 0 & & & h_N \end{pmatrix}, \tag{91}$$

where $h_0, \dots, h_N$ are the scaling coefficients corresponding to ϕ. By a suitable elimination we will successively reduce the upper and lower bands from $\mathbf{A}_j$ such that after i steps we obtain

$$\mathbf{A}_j^{(i)} = \begin{pmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \\ h^{(i)}_{\lceil \frac{i}{2} \rceil} & 0 & & \\ h^{(i)}_{\lceil \frac{i}{2} \rceil + 1} & 0 & & \\ \vdots & h^{(i)}_{\lceil \frac{i}{2} \rceil} & & \\ \vdots & \vdots & & \\ h^{(i)}_{N - \lfloor \frac{i}{2} \rfloor} & & & \\ 0 & & & \\ \vdots & & & \\ & & & h^{(i)}_{N - \lfloor \frac{i}{2} \rfloor} \\ & & & 0 \\ & & & \vdots \\ 0 & & & 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} \lceil \frac{i}{2} \rceil \\ \\ \\ \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} \lfloor \frac{i}{2} \rfloor \end{matrix}, \quad \mathbf{A}_j^{(0)} := \mathbf{A}_j. \tag{92}$$

In [16], it was proved for B-spline scaling functions that

$$h^{(i)}_{\lceil i/2\rceil},\dots,h_{N-\lfloor i/2\rfloor}\neq 0,\quad i=1,\dots,N. \tag{93}$$

Therefore, the ellimination is always possible. The elimination matrices are of the form

$$H_j^{(2i-1)} := \operatorname{diag}\left(\mathbf{I}_{i-1},\mathbf{U}_{2i-1},\dots,\mathbf{U}_{2i-1},\mathbf{I}_{N-1}\right), \tag{94}$$

$$H_j^{(2i)} := \operatorname{diag}\left(\mathbf{I}_{N-i},\mathbf{L}_{2i},\dots,\mathbf{L}_{2i},\mathbf{I}_{i-1}\right), \tag{95}$$

where

$$\mathbf{U}_{i+1} := \begin{pmatrix} 1-\frac{h^{(i)}_{\lceil i/2\rceil}}{h^{(i)}_{\lceil i/2\rceil+1}} \\ 0 & 1\end{pmatrix},\quad \mathbf{L}_{i+1} := \begin{pmatrix} 1 & 0 \\ -\frac{h^{(i)}_{N-\lfloor i/2\rfloor}}{h^{(i)}_{N-\lfloor i/2\rfloor-1}} & 1\end{pmatrix}. \tag{96}$$

It is easy to see that indeed

$$\mathbf{A}_j^{(i)} = \mathbf{H}_j^{(i)}\mathbf{A}_j^{(i-1)}. \tag{97}$$

After N elimination steps we obtain the matrix $\mathbf{A}_j^{(N)}$ which looks as follows

$$\mathbf{A}_j^{(N)} = \mathbf{H}_j\mathbf{A}_j = \begin{pmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \\ b & 0 & & \\ 0 & 0 & & \\ 0 & b & & \\ \vdots & 0 & \ddots & \\ & & & b \\ & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} \lceil \frac{N}{2}\rceil \\ \\ \\ \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} \lfloor \frac{N}{2}\rfloor \end{matrix},\quad \text{where}\quad \mathbf{H}_j := \mathbf{H}_j^{(N)}\dots\mathbf{H}_j^{(1)}, \tag{98}$$

with $b\neq 0$. We define

$$\mathbf{B}_j := \left(\mathbf{A}_j^{(N)}\right)^{-1} = \begin{pmatrix} \underbrace{0 \dots 0}_{\lceil \frac{N}{2}\rceil} & b^{-1} & 0 & 0 & 0 & \dots & & & 0 \\ 0 \dots 0 & 0 & 0 & b^{-1} & 0 & \dots & & & 0 \\ & & & & & \ddots & & & \\ & & & & & & b^{-1} & \underbrace{0 \dots 0}_{\lfloor \frac{N}{2}\rfloor} \end{pmatrix} \tag{99}$$

and

$$\mathbf{F}_j := \begin{pmatrix} 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \\ 1 & 0 & \\ 0 & 0 & \\ 0 & 1 & \\ \vdots & 0 & \ddots & \\ & & & 1 \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} \lceil \frac{N}{2}\rceil - 1 \\ \\ \\ \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} \lfloor \frac{N}{2}\rfloor + 1 \end{matrix}. \tag{100}$$

Then, we have

$$\mathbf{B}_j\mathbf{F}_j = \mathbf{0}. \tag{101}$$

After these preparations we define extended versions of the matrices $\mathbf{H}_j$, $\mathbf{A}_j$, $\mathbf{A}_j^{(N)}$, and $\mathbf{B}_j$ by

$$\hat{\mathbf{H}}_j := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{H}_j & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \quad \hat{\mathbf{A}}_j^{(N)} := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{A}_j^{(N)} & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \tag{102}$$

$$\hat{\mathbf{A}}_j := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{A}_j & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \quad \hat{\mathbf{B}}_j^T := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{B}_j^T & \\ & & \mathbf{I}_{N-1} \end{pmatrix}. \tag{103}$$

Note that $\hat{\mathbf{H}}_j$, $\hat{\mathbf{A}}_j$, $\hat{\mathbf{A}}_j^{(N)}$, and $\hat{\mathbf{B}}_j$ are all matrices of the size $(\#\mathscr{I}_{j+1}) \times (\#\mathscr{I}_j)$. Hence, the completion of $\hat{\mathbf{A}}_j^{(N)}$ has to be a $(\#\mathscr{I}_{j+1}) \times 2^j$. On the contrary to the construction in [16], we define an expanded version of $\mathbf{F}_j$ as in [5], because it leads to a more natural formulation, when the entries of both the refinement matrices belong to $\sqrt{2}\mathbb{Q}$. The difference is in multiplication by $\sqrt{2}$,

$$\hat{\mathbf{F}}_j := \sqrt{2} \left(\begin{array}{c|c|c} \multicolumn{3}{c}{\mathbf{O}} \\ \hline \mathbf{I}_{\lceil \frac{N}{2} \rceil - 1} & & \\ \cline{1-1} & \mathbf{F}_j & \\ \cline{3-3} & & \mathbf{I}_{\lfloor \frac{N}{2} \rfloor} \\ \hline \multicolumn{3}{c}{\mathbf{O}} \end{array} \right) \begin{array}{l} \} N-1 \\ \\ \\ \\ \} N-1 \end{array} . \tag{104}$$

The above findings can be summarized as follows.

Lemma 12. *The following relations hold:*

$$\hat{\mathbf{B}}_j \hat{\mathbf{A}}_j^{(N)} = \mathbf{I}_{\#\mathscr{I}_j}, \quad \frac{1}{2} \hat{\mathbf{F}}_j^T \hat{\mathbf{F}}_j = \mathbf{I}_{2^j} \tag{105}$$

and

$$\hat{\mathbf{B}}_j \hat{\mathbf{F}}_j = \mathbf{0}, \quad \hat{\mathbf{F}}_j^T \hat{\mathbf{A}}_j^{(N)} = \mathbf{0}. \tag{106}$$

The proof of this lemma is similar to the proof in [16]. Note the refinement matrix $\mathbf{M}_{j,0}$ can be factorized as

$$\mathbf{M}_{j,0} = \mathbf{P}_j \hat{\mathbf{A}}_j = \mathbf{P}_j \hat{\mathbf{H}}_j^{-1} \hat{\mathbf{A}}_j^{(N)} \tag{107}$$

with

$$\mathbf{P}_j := \left(\begin{array}{c|c|c} \mathbf{M}_L & & \\ \cline{1-1} & \mathbf{I}_{\#\mathscr{I}_{j+1-2N}} & \\ \cline{3-3} & & \mathbf{M}_R \end{array} \right). \tag{108}$$

Now we are able to define the initial stable completions of the refinement matrices $\mathbf{M}_{j,0}$.

Lemma 13. *Under the above assumptions, the matrices*

$$\check{\mathbf{M}}_{j,1} := \mathbf{P}_j \hat{\mathbf{H}}_j^{-1} \hat{\mathbf{F}}_j, \quad j \geq j_0, \tag{109}$$

are uniformly stable completions of the matrices $\mathbf{M}_{j,0}$. *Moreover, the inverse*

$$\check{\mathbf{G}}_j = \begin{pmatrix} \check{\mathbf{G}}_{j,0} \\ \check{\mathbf{G}}_{j,1} \end{pmatrix} \tag{110}$$

of $\check{\mathbf{M}}_j = \left(\mathbf{M}_{j,0}, \check{\mathbf{M}}_{j,1} \right)$ *is given by*

$$\check{\mathbf{G}}_{j,0} = \hat{\mathbf{B}}_j \hat{\mathbf{H}}_j \mathbf{P}_j^{-1}, \quad \check{\mathbf{G}}_{j,1} = \frac{1}{2} \hat{\mathbf{F}}_j^T \hat{\mathbf{H}}_j \mathbf{P}_j^{-1}. \tag{111}$$

The proof of this lemma is straightforward and similar to the proof in [16]. Then using the initial stable completion $\check{\mathbf{M}}_{j,1}$ we are already able to contruct wavelets according to the Theorem 11.

7 Norm equivalences

In this section, we prove norm equivalences and we show that Ψ and $\tilde{\Psi}$ are Riesz bases for the space $L^2([0,1])$. Furthermore, we show that $\left\{2^{-s|\lambda|}\psi_\lambda, \lambda \in \mathscr{J}\right\}$ is a Riesz basis for Sobolev space $H^s([0,1])$ for some s specified below. The proofs are based on the theory developed in [13] and [16].

Let us define

$$\gamma := \sup\left\{s : \phi \in H^s(\mathbb{R})\right\}, \quad \tilde{\gamma} := \sup\left\{s : \tilde{\phi} \in H^s(\mathbb{R})\right\}. \tag{112}$$

It is known that $\gamma = N - \frac{1}{2}$. The Sobolev exponent of smoothness $\tilde{\gamma}$ can be computed by the method from [21]. The functions in Φ_j and Ψ_j, $j \geq j_0$, have the Sobolev regularity at least γ, because the primal scaling functions are B-splines and the primal wavelets are finite linear combinations of the primal scaling functions. Similarly, the functions in $\tilde{\Phi}_j$ and $\tilde{\Psi}_j$, $j \geq j_0$, have the Sobolev regularity at least $\tilde{\gamma}$.

Theorem 14. *i) The sets* $\left\{\Phi_j\right\} := \left\{\Phi_j\right\}_{j \geq j_0}$ *and* $\left\{\tilde{\Phi}_j\right\} := \left\{\tilde{\Phi}_j\right\}_{j \geq j_0}$ *are uniformly stable, i.e.*

$$c \left\|b\right\|_{l_2(\mathscr{I}_j)} \leq \left\|\sum_{k \in \mathscr{I}_j} b_k \phi_{j,k}\right\| \leq C \left\|b\right\|_{l_2(\mathscr{I}_j)} \quad \text{for all } b = \left\{b_k\right\}_{k \in \mathscr{I}_j} \in l^2(\mathscr{I}_j), \; j \geq j_0. \tag{113}$$

ii) For all $j \geq j_0$*, the Jackson inequalities hold, i.e.*

$$\inf_{v_j \in S_j} \left\|v - v_j\right\| \lesssim 2^{-sj} \left\|v\right\|_{H^s([0,1])} \quad \text{for all } v \in H^s([0,1]) \text{ and } s < N, \tag{114}$$

and

$$\inf_{v_j \in \tilde{S}_j} \left\|v - v_j\right\| \lesssim 2^{-sj} \left\|v\right\|_{H^s([0,1])} \quad \text{for all } v \in H^s([0,1]) \text{ and } s < \tilde{N}. \tag{115}$$

iii) For all $j \geq j_0$*, the Bernstein inequalities hold, i.e.*

$$\left\|v_j\right\|_{H^s([0,1])} \lesssim 2^{sj} \left\|v_j\right\| \quad \text{for all } v_j \in S_j \text{ and } s < \gamma, \tag{116}$$

and

$$\left\|v_j\right\|_{H^s([0,1])} \lesssim 2^{sj} \left\|v_j\right\| \quad \text{for all } v_j \in \tilde{S}_j \text{ and } s < \tilde{\gamma}. \tag{117}$$

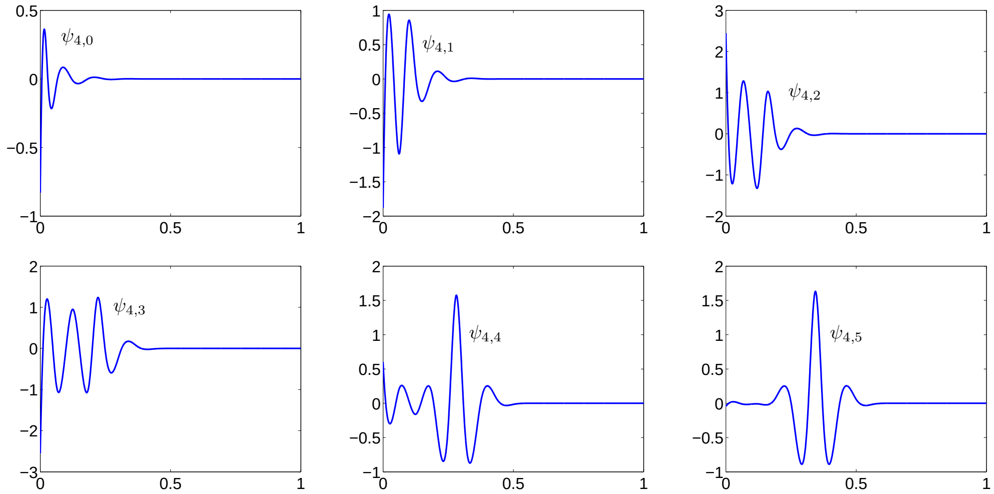

Fig. 5 Some primal wavelets for $N = 4$ and $\tilde{N} = 6$ without boundary conditions.

Proof i) Due to Lemma 2.1 in [16], the collections $\{\boldsymbol{\Phi}_j\} := \{\boldsymbol{\Phi}_j\}_{j \geq j_0}$ and $\{\tilde{\boldsymbol{\Phi}}_j\} := \{\tilde{\boldsymbol{\Phi}}_j\}_{j \geq j_0}$ are uniformly stable, if $\boldsymbol{\Phi}_j$ and $\tilde{\boldsymbol{\Phi}}_j$ are biorthogonal,

$$\left\|\phi_{j,k}\right\| \lesssim 1, \left\|\tilde{\phi}_{j,k}\right\| \lesssim 1, \quad k \in \mathscr{I}_j,\ j \geq j_0, \tag{118}$$

and $\boldsymbol{\Phi}_j$ and $\tilde{\boldsymbol{\Phi}}_j$ are locally finite, i.e.

$$\#\left\{k' \in \mathscr{I}_j : \Omega_{j,k'} \cap \Omega_{j,k} \neq \emptyset\right\} \lesssim 1, \quad \text{for all } k \in \mathscr{I}_j,\ j \geq j_0, \tag{119}$$

and

$$\#\left\{k' \in \mathscr{I}_j : \tilde{\Omega}_{j,k'} \cap \tilde{\Omega}_{j,k} \neq \emptyset\right\} \lesssim 1, \quad \text{for all } k \in \mathscr{I}_j,\ j \geq j_0, \tag{120}$$

where $\Omega_{j,k} := \text{supp } \phi_{j,k}$ and $\tilde{\Omega}_{j,k} := \text{supp } \tilde{\phi}_{j,k}$. By (40) the sets $\boldsymbol{\Phi}_j$ and $\tilde{\boldsymbol{\Phi}}_j$ are biorthogonal. The properties (118), (119), and (120) follow from (15), (21), and (35).

ii) By Lemma 2.1 in [16], the Jackson inequalities are the consequences of i) and the polynomial exactness (17) and (68).

iii) The Bernstein inequalities follow from i) and the regularity of basis functions, for details see [14].

The following fact follows from [13].

Corollary 1. *We have the norm equivalences*

$$\|v\|_{H^s}^2 \sim 2^{2sj_0} \left\| \sum_{k \in \mathscr{I}_{j_0}} \left\langle v, \tilde{\phi}_{j_0,k} \right\rangle \phi_{j_0,k} \right\|^2 + \sum_{j=j_0}^{\infty} 2^{2sj} \left\| \sum_{k \in \mathscr{J}_j} \left\langle v, \tilde{\psi}_{j,k} \right\rangle \psi_{j,k} \right\|^2, \tag{121}$$

where $v \in H^s([0,1])$ *and* $s \in (-\tilde{\gamma}, \gamma)$.

The norm equivalence for $s = 0$, Theorem 11, and Lemma 13, imply that

$$\boldsymbol{\Psi} := \boldsymbol{\Phi}_{j_0} \cup \bigcup_{j=j_0}^{\infty} \boldsymbol{\Psi}_j \quad \text{and} \quad \tilde{\boldsymbol{\Psi}} := \tilde{\boldsymbol{\Phi}}_{j_0} \cup \bigcup_{j=j_0}^{\infty} \tilde{\boldsymbol{\Psi}}_j \tag{122}$$

are biorthogonal Riesz bases of the space $L^2([0,1])$. Let us define

$$\mathbf{D} = \left(\mathbf{D}_{\lambda,\tilde{\lambda}}\right)_{\lambda,\tilde{\lambda} \in \mathscr{J}}, \quad \mathbf{D}_{\lambda,\tilde{\lambda}} := \delta_{\lambda,\tilde{\lambda}} 2^{|\lambda|}, \quad \lambda, \tilde{\lambda} \in \mathscr{J}. \tag{123}$$

The relation (121) implies that $\mathbf{D}^{-s}\boldsymbol{\Psi}$ is a Riesz basis of the Sobolev space $H^s([0,1])$ for $s \in (-\tilde{\gamma}, \gamma)$.

8 Adaptation to Complementary Boundary Conditions

In this section, we introduce a construction of well-conditioned spline-wavelet bases on the interval satisfying complementary boundary conditions of the first order. This means that the primal wavelet basis is adapted to homogeneous Dirichlet boundary conditions of the first order, whereas the dual wavelet basis preserves the full degree of polynomial exactness. This construction is based on the spline-wavelet bases constructed above. As already mentioned in Remark 7, in the linear case, i.e. $N = 2$, our bases are identical to the bases constructed in [24]. The adaptation of these bases to complementary boundary conditions can be found in [24]. Thus, we consider only the case $N \geq 3$.

Let $\boldsymbol{\Phi}_j = \left\{\phi_{j,k}, k = -N+1, \ldots, 2^j - 1\right\}$ be defined as above. Note that the functions $\phi_{j,-N+1}$, $\phi_{j,2^j-1}$ are the only two functions which do not vanish at zero. Therefore, defining

$$\boldsymbol{\Phi}_j^{comp} = \left\{\phi_{j,k}, k = -N+2, \ldots, 2^j - 2\right\} \tag{124}$$

we obtain the primal scaling bases satisfying complementary boundary conditions of the first order.

On the dual side, we also need to omit one scaling function at each boundary, because the number of the primal scaling functions must be the same as the number of the dual scaling functions. Let $\Theta_j = \left\{\theta_{j,k}, k \in \mathscr{I}_j\right\}$ be the dual scaling basis on the level j before biorthogonalization from Section 4. There are the boundary functions of two types. Recall that the functions $\theta_{j,-N+1}, \ldots, \theta_{j,-N+\tilde{N}}$ are the left boundary functions of the first type which are defined to preserve polynomial exactness of the order $\tilde{N}$. The functions $\theta_{j,-N+\tilde{N}+1}, \ldots, \theta_{j,\tilde{N}-2}$ are the left boundary functions of the second type. The right boundary scaling functions are then derived by the reflection of the left boundary functions. Since we want to preserve the full degree of polynomial exactness, we omit one function of the second type at each boundary. Thus, we define

$$\theta_{j,k}^{comp} = \begin{cases} \theta_{j,k-1}, & k = -N+2, \ldots, -N+\tilde{N}+1, \\ \theta_{j,k}, & k = -N+\tilde{N}+2, \ldots, 2^j - \tilde{N} - 2, \\ \theta_{j,k+1}, & k = 2^j - \tilde{N} - 1, \ldots, 2^j - 2. \end{cases} \tag{125}$$

Since the set $\Theta_j^{comp} := \left\{ \theta_{j,k}^{comp} : k = -N+2, \ldots, 2^j - 2 \right\}$ is not biorthogonal to Φ_j, we derive a new set $\tilde{\Phi}_j^{comp}$ from Θ_j^{comp} by biorthogonalization. Let $\mathbf{Q}_j^{comp} = \left(\left\langle \phi_{j,k}, \theta_{j,l}^{comp} \right\rangle \right)_{k,l=-N+2}^{2^j-2}$, then viewing $\tilde{\Phi}_j^{comp}$ and Θ_j^{comp} as column vectors we define

$$\tilde{\Phi}_j^{comp} := \left(\mathbf{Q}_j^{comp} \right)^{-T} \Theta_j^{comp}. \tag{126}$$

Our next goal is to determine the corresponding wavelets

$$\Psi_j^{comp} := \left\{ \psi_{j,k}^{comp}, k = 0, \ldots, 2^j - 1 \right\}, \quad \tilde{\Psi}_j^{comp} := \left\{ \tilde{\psi}_{j,k}^{comp}, k = 0, \ldots, 2^j - 1 \right\}. \tag{127}$$

It can be done by the method of a stable completion as in Section 6.

9 Quantitative Properties of Constructed Bases

In this section the condition numbers of the scaling bases, the single-scale wavelet bases and the multiscale wavelet bases are computed. As in [24] it can be improved by the L^2-normalization on the primal side. It will be shown that in the case of cubic spline wavelets bases the construction presented in this paper yields optimal L^2-stability, which is not the case of constructions in [16] and [24]. The condition numbers of the scaling bases and the wavelet bases satisfying the complementary boundary conditions of the first order are presented as well. The other criteria for the effectiveness of the wavelet bases is the condition number of the corresponding preconditioned stiffness matrix. To improve it further we apply an orthogonal transformation to the scaling basis on the coarsest level and then we use a diagonal matrix for preconditioning.

It is known that Riesz bounds (2) of the basis Φ_j can be computed by

$$c = \sqrt{\lambda_{min}(\mathbf{G}_j)}, \quad C = \sqrt{\lambda_{max}(\mathbf{G}_j)}, \tag{128}$$

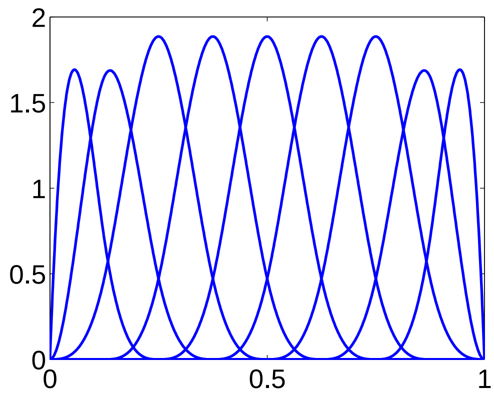

Fig. 6 Primal scaling functions for $N = 4$ and $j = 3$ satisfying complementary boundary conditions of the first order.

where $\mathbf{G}_j$ is the Gram matrix, i.e. $\mathbf{G}_j = \left(\langle \phi_{j,k}, \phi_{j,l} \rangle\right)_{k,l \in \mathscr{I}_j}$, and $\lambda_{min}(\mathbf{G}_j)$, $\lambda_{max}(\mathbf{G}_j)$ denote the smallest and the largest eigenvalue of $\mathbf{G}_j$, respectively. The Riesz bounds of $\tilde{\Phi}_j$, Ψ_j and $\tilde{\Psi}_j$ are computed in a similar way.

The condition numbers of the constructed bases are presented in Table 2. To improve it further we provide a diagonal rescaling in the following way:

$$\phi_{j,k}^N = \frac{\phi_{j,k}}{\sqrt{\langle \phi_{j,k}, \phi_{j,k} \rangle}}, \quad \tilde{\phi}_{j,k}^N = \tilde{\phi}_{j,k} * \sqrt{\langle \phi_{j,k}, \phi_{j,k} \rangle}, \quad k \in \mathscr{I}_j, \quad j \geq j_0, \tag{129}$$

$$\psi_{j,k}^N = \frac{\psi_{j,k}}{\sqrt{\langle \psi_{j,k}, \psi_{j,k} \rangle}}, \quad \tilde{\psi}_{j,k}^N = \tilde{\psi}_{j,k} * \sqrt{\langle \psi_{j,k}, \psi_{j,k} \rangle}, \quad k \in \mathscr{J}_j, \quad j \geq j_0. \tag{130}$$

Then the new primal scaling and wavelet bases are normalized with respect to the L^2-norm. As already mentioned in Remark 7, the resulting bases for $N = 2$ are the same as those designed in [24] and [25]. For the quadratic spline-wavelet bases, i.e. $N = 3$, the condition of our bases is comparable to the condition of the bases from [24] and [25]. In [3], it was shown that for any spline wavelet basis of order N on the real line, the condition is bounded below by 2^{N-1}. This result readily carries over to the case of spline wavelet bases on the interval. Now, the constructions from [24], [25] yields the wavelet bases whose Riesz bounds are nearly optimal, i.e. cond $\Psi_j^N \approx 2^{N-1}$ for $N = 2$ and $N = 3$. Unfortunately, the L^2-stability gets considerably worse for $N \geq 4$. As can be seen in Table 2, the column "Ψ_j^N", the presented construction seems to yield the optimal L^2-stability also for $N = 4$. Note that the cases $N = 4$, $\tilde{N} = 4$ and $N = 5$, $\tilde{N} < 9$ are not included in Table 2. It was shown in [9] that the corresponding scaling functions and wavelets do not belong to the space L^2.

In Table 3 the condition of the multiscale wavelet bases $\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j$ is presented.

It is known that the condition number of the original basis on the real line from [9] is less than or equal to the condition number of the interval wavelet basis where the inner functions are identical to the basis functions from [9]. Therefore, we use the condition number of the wavelet bases from [9] as a benchmark. In Table 4, we compare the condition number of our wavelet bases and the wavelet bases from [9], [24].

In case $N = 5$, the condition numbers of the scaling bases and the single-scale wavelet bases seem to be optimal, but the condition numbers of the multiscale wavelet bases are not close to the condition numbers of the corresponding wavelet bases on the real line. However, in comparison with [24] the condition number is significantly improved for $N = 5$ and $\tilde{N} = 9$. Therefore the construction of well-conditioned high-order biorthogonal spline wavelets is still an open problem.

The condition of the single-scale bases adapted to complementary boundary condition of the first order are listed in Table 5. We improve the condition of the constructed bases by the L^2-normalization. For $N = 4$ the

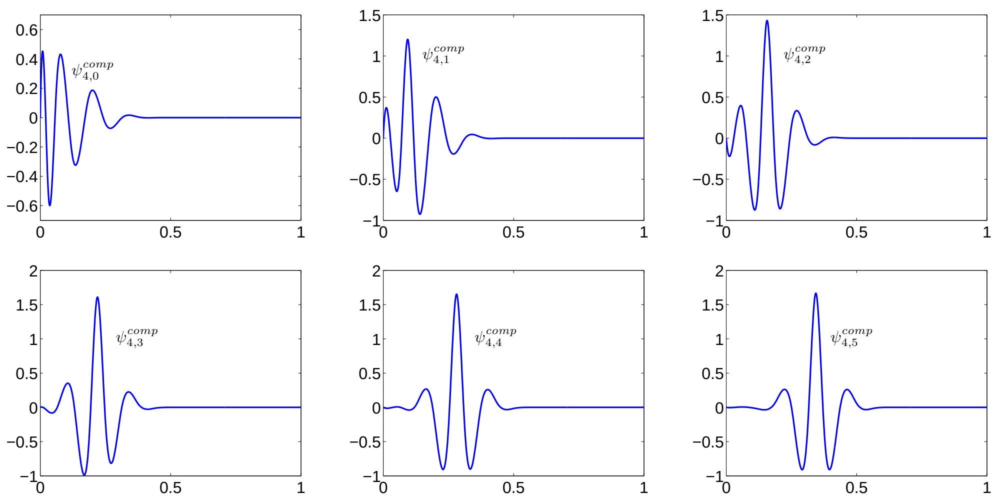

Fig. 7 Some primal wavelets for $N = 4$ and $\tilde{N} = 6$ satisfying the complementary boundary conditions of the first order.

condition number of the bases constructed in this paper is again significantly better than the condition of the bases from [24].

The other criteria for the effectiveness of a wavelet basis is the condition number of the corresponding stiffness matrix. Here, let us consider the stiffness matrix for the Poisson equation:

$$\mathbf{A}_{j_0,s} = \left(\left\langle \left(\psi_{j,k}^{comp} \right)', \left(\psi_{l,m}^{comp} \right)' \right\rangle \right)_{\psi_{j,k}^{comp}, \psi_{l,m}^{comp} \in \Psi_{j_0,s}^{comp}}, \tag{131}$$

where $\Psi_{j_0,s}^{comp} = \Phi_{j_0}^{comp} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j^{comp}$ denotes the multiscale basis adapted to complementary boundary conditions. It is well-known that the condition number of $\mathbf{A}_{j_0,s}$ increases quadratically with the matrix size. To remedy

Table 2 The condition of single-scale scaling and wavelet bases

N	$\tilde{N}$	j	Φ_j	Φ_j^N	$\tilde{\Phi}_j$	$\tilde{\Phi}_j^N$	Ψ_j	Ψ_j^N	$\tilde{\Psi}_j$	$\tilde{\Psi}_j^N$
2	2	10	2.00	1.73	2.30	1.97	2.00	2.00	2.02	2.00
2	4	10	2.00	1.73	2.09	1.80	2.00	2.00	2.04	2.00
2	6	10	2.00	1.73	2.26	2.03	2.00	2.00	2.30	2.26
2	8	10	2.00	1.73	2.90	2.78	2.34	2.22	3.14	3.81
3	3	10	3.25	2.76	7.58	6.37	4.49	4.00	7.07	4.27
3	5	10	3.25	2.76	3.93	3.49	4.63	4.00	5.55	4.05
3	7	10	3.25	2.76	3.53	3.11	4.55	4.00	5.13	4.01
3	9	10	3.25	2.76	3.75	3.32	4.44	4.00	5.51	4.23
4	6	10	5.18	4.42	10.88	9.07	14.02	8.00	24.36	9.23
4	8	10	5.18	4.42	6.69	5.88	13.96	8.00	16.98	8.20
4	10	10	5.18	4.42	5.83	5.16	13.82	8.00	15.27	8.00
5	9	10	8.32	7.13	29.87	25.23	67.74	27.44	169.76	68.90
5	11	10	8.32	7.13	12.10	11.74	16.00	16.00	45.12	21.65
5	13	10	8.32	7.13	28.49	45.60	16.00	16.00	22.64	22.23

Table 3 The condition of the multiscale wavelet bases

N	$\tilde{N}$	j_0	$\Psi_{j_0,1}^N$	$\Psi_{j_0,2}^N$	$\Psi_{j_0,3}^N$	$\Psi_{j_0,4}^N$	$\Psi_{j_0,5}^N$	$\tilde{\Psi}_{j_0,1}^N$	$\tilde{\Psi}_{j_0,2}^N$	$\tilde{\Psi}_{j_0,3}^N$	$\tilde{\Psi}_{j_0,4}^N$	$\tilde{\Psi}_{j_0,5}^N$
2	2	2	1.98	2.27	2.52	2.65	2.76	2.20	2.42	2.65	2.78	2.87
2	4	3	2.13	2.25	2.30	2.33	2.34	2.15	2.26	2.31	2.33	2.35
2	6	4	2.47	2.71	2.84	2.92	2.99	2.60	2.78	2.88	2.94	3.00
2	8	4	3.71	4.77	5.35	5.68	5.89	4.44	5.17	5.57	5.82	5.98
3	3	3	4.92	6.01	7.15	7.87	8.50	7.25	8.54	9.50	10.08	10.48
3	5	4	4.51	4.82	5.01	5.10	5.14	4.63	4.98	5.11	5.15	5.16
3	7	4	4.19	4.38	4.44	4.46	4.48	4.24	4.39	4.45	4.48	4.49
3	9	5	4.44	4.55	4.61	4.64	4.65	4.48	4.58	4.62	4.64	4.66
4	6	4	9.55	10.90	11.88	12.50	12.90	10.88	12.90	13.35	13.48	13.58
4	8	5	8.01	8.31	8.54	8.68	8.76	8.23	8.60	8.73	8.79	8.81
4	10	5	7.89	8.02	8.09	8.12	8.13	7.93	8.05	8.11	8.13	8.14
5	9	5	30.22	64.60	75.17	81.03	84.81	72.34	83.19	87.93	90.11	91.27
5	11	5	84.40	631.61	3004.08	$> 10^4$	$> 10^4$	54.08	401.23	3004.08	$> 10^4$	$> 10^4$

Table 4 The condition number of our multiscale wavelet bases $\Psi_{j_0,5}^N$ and $\tilde{\Psi}_{j_0,5}^N$ and multiscale wavelet bases from [9] and [24]

N	$\tilde{N}$	j_0	s	$\Psi_{j_0,5}^{CDF}$	$\Psi_{j_0,5}^{Primbs}$	$\Psi_{j_0,5}^N$	$\tilde{\Psi}_{j_0,5}^{CDF}$	$\tilde{\Psi}_{j_0,5}^{Primbs}$	$\tilde{\Psi}_{j_0,5}^N$
3	3	3	5	6.68	6.25	8.50	8.52	8.17	10.48
3	5	4	5	4.36	5.31	5.14	4.37	5.36	5.16
3	7	4	5	4.04	8.57	4.48	4.04	8.63	4.49
3	9	5	5	4.00	25.40	4.65	4.00	25.76	4.66
4	6	4	5	9.89	141.95	12.90	10.43	160.54	13.58
4	8	5	5	8.27	257.41	8.76	8.27	258.56	8.81
4	10	5	5	8.04	917.10	8.13	8.04	935.38	8.14
4	12	5	5	8.01	3971.65	8.44	8.01	3992.29	8.45
5	9	5	5	17.64	$> 10^4$	84.81	18.01	$> 10^4$	91.27

this, we use the diagonal matrix for preconditioning

$$\mathbf{A}_{j_0,s}^{prec} = \mathbf{D}_{j_0,s}^{-1}\mathbf{A}_{j_0,s}\mathbf{D}_{j_0,s}^{-1}, \quad \mathbf{D}_{j_0,s} = \mathrm{diag}\left(\left\langle \left(\psi_{j,k}^{comp}\right)', \left(\psi_{j,k}^{comp}\right)' \right\rangle^{1/2}\right)_{\psi_{j,k}^{comp}\in\Psi_{j_0,s}^{comp}}. \tag{132}$$

To improve further the condition number of $\mathbf{A}_{j_0,s}^{prec}$ we apply the orthogonal transformation to the scaling basis on the coarsest level as in [7] and then we use the diagonal matrix for preconditioning. We denote the obtained matrix by $\mathbf{A}_{j_0,s}^{ort}$. The condition numbers of the resulting matrices are listed in Table 6.

10 Adaptive wavelet methods

In recent years adaptive wavelet methods have been successfully used for solving partial differential as well as integral equations, both linear and nonlinear. It has been shown that these methods converge and that they are asymptotically optimal in the sense that a storage and a number of floating point operations, needed to resolve the problem with desired accuracy, remain proportional to the problem size when the resolution of the discretization is refined. Thus, the computational complexity for all steps of the algorithm is controlled.

The effectiveness of adaptive wavelet methods is strongly influenced by the choice of a wavelet basis, in particular by the condition of the basis. In this section, our intention is to compare the quantitative behaviour of the adaptive wavelet method for the cubic spline wavelet bases constructed in this paper and the cubic spline wavelet bases from [24].

Example 15. We consider the one-dimensional Poisson equation with homogeneous Dirichlet boundary conditions

$$-u'' = f, \quad \text{in} \quad \Omega = (0,1), \quad u(0) = u(1) = 0, \tag{133}$$

Table 5 The condition of scaling bases and single-scale wavelet bases satisfying complementary boundary conditions of the first order

N	$\tilde{N}$	j	Φ_j	Φ_j^N	$\tilde{\Phi}_j$	$\tilde{\Phi}_j^N$	Ψ_j	Ψ_j^N	$\tilde{\Psi}_j$	$\tilde{\Psi}_j^N$
3	3	10	2.74	2.74	4.49	4.34	4.00	4.00	4.13	4.00
3	5	10	2.74	2.74	4.94	4.58	4.00	4.00	6.68	6.27
3	7	10	2.74	2.74	8.61	8.33	4.84	4.27	12.11	16.05
3	9	10	2.74	2.74	17.94	17.78	8.16	6.25	25.17	46.10
4	6	10	4.53	4.31	7.90	6.83	9.47	8.00	16.46	8.00
4	8	10	4.53	4.31	11.16	10.04	8.46	8.03	25.40	15.32
4	10	10	4.53	4.31	17.90	16.97	8.39	8.42	37.78	35.93
5	9	10	7.58	6.89	15.81	13.85	35.01	16.02	80.84	33.60
5	11	10	7.58	6.89	29.00	26.39	16.00	16.00	132.90	74.70
5	13	10	7.58	6.89	289.13	440.54	118.19	89.12	720.32	5884.77

Table 6 The condition number of the stiffness matrices $\mathbf{A}_{j,s}^{prec}$, $\mathbf{A}_{j,s}^{ort}$ of the size $M \times M$

N	$\tilde{N}$	j	s	M	$\mathbf{A}_{j,s}^{prec}$	$\mathbf{A}_{j,s}^{ort}$	N	$\tilde{N}$	j	s	M	$\mathbf{A}_{j,s}^{prec}$	$\mathbf{A}_{j,s}^{ort}$
3	3	3	1	16	12.24	3.78	4	6	4	1	33	48.98	15.25
			4	128	12.82	5.05				4	259	51.61	16.15
			7	1024	12.86	5.37				7	2049	50.28	16.31
3	5	4	1	32	52.97	4.20	4	8	5	1	65	205.56	15.92
			4	256	55.09	8.41				4	513	208.88	26.80
			7	2048	55.24	9.47				7	4097	209.31	27.69
3	7	4	1	32	71.07	10.74	5	7	5	1	66	183.57	159.08
			4	256	71.90	33.52				4	514	214.27	214.40
			7	2048	71.91	38.66				7	4098	222.57	222.62
4	4	4	1	33	47.02	15.38	5	9	5	1	66	191.19	171.91
			4	259	50.01	18.13				4	514	225.92	225.69
			7	2049	50.28	18.91				7	4098	233.24	233.24

whose solution u is given by

$$u(x) = 4\frac{e^{50x}-1}{e^{50}-1}\left(1-\frac{e^{50x}-1}{e^{50}-1}\right)+x(1-x), \quad x \in \Omega. \tag{134}$$

The solution exhibits steep a gradient near the boundary, see Figure 8. Let us define the diagonal matrix

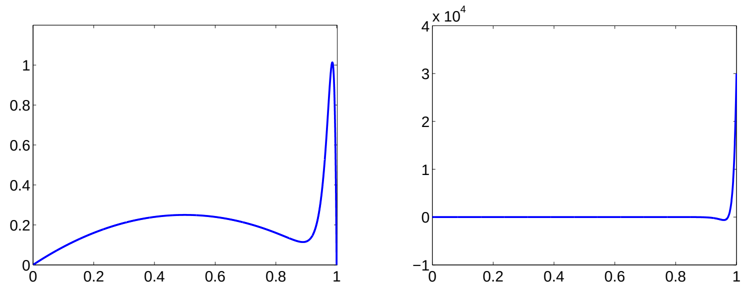

Fig. 8 The exact solution and the right hand side of (133).

$$\mathbf{D} = \operatorname{diag}\left(\left\langle \psi'_{j,k}, \psi'_{j,k}\right\rangle^{1/2}\right)_{\psi_{j,k}\in\Psi} \tag{135}$$

and operators

$$\mathbf{A} = \mathbf{D}^{-1}\left\langle \Psi', \Psi'\right\rangle \mathbf{D}^{-1}, \quad \mathbf{f} = \mathbf{D}^{-1}\left\langle f, \Psi\right\rangle. \tag{136}$$

Then the variational formulation of (133) is equivalent to

$$\mathbf{AU} = \mathbf{f} \tag{137}$$

and the solution u is given by $u = \mathbf{U}\mathbf{D}^{-1}\Psi$. We solve the infinite dimensional problem (137) by the inexact damped Richardson iterations. This algorithm was originally proposed by Cohen, Dahmen and DeVore in [10]. Here, we use a modified version from [30].

Figure 9 shows a convergence history for the spline-wavelet bases designed in this contribution with $N = 4$ and $\tilde{N} = 6$ denoted by CF and the spline-wavelet bases with the same polynomial exactness from [24]. We use also the algorithm with the stiffness matrix $\mathbf{A}^{ort}$ which has lower condition number, see Table 6. Its convergence history is denoted by CFort. Note that the relative error in the energy norm for an adaptive scheme with our bases is significantly smaller even though the number of the involved basis functions is about half compared with the bases in [24].

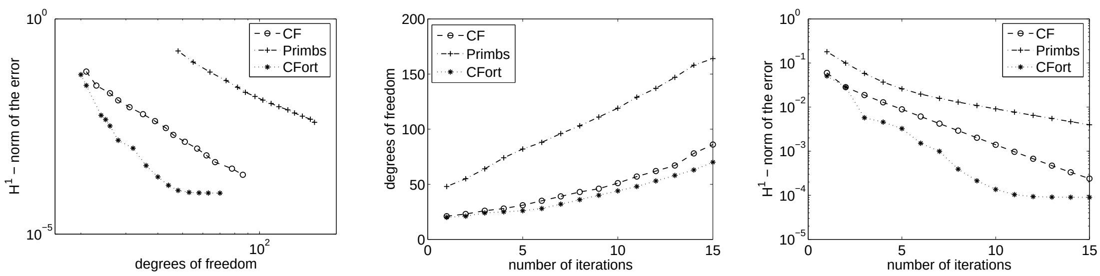

Fig. 9 Convergence history for 1d example, comparison of our wavelet bases with and without orthogonalization and wavelet bases from [24].

Example 16. We consider the two-dimensional Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega = (0,1)^2, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{138}$$

with the solution u given by

$$u(x,y) = u(x)\,u(y), \quad (x,y) \in \Omega, \tag{139}$$

where $u(x)$, $u(y)$ are given by (134). We use the adaptive wavelet scheme with the cubic wavelet basis adapted to homogeneous Dirichlet boundary conditions of the first order. The convergence history for our wavelet bases with and without orthogonalization and wavelet bases from [24] is shown in Figure 10.

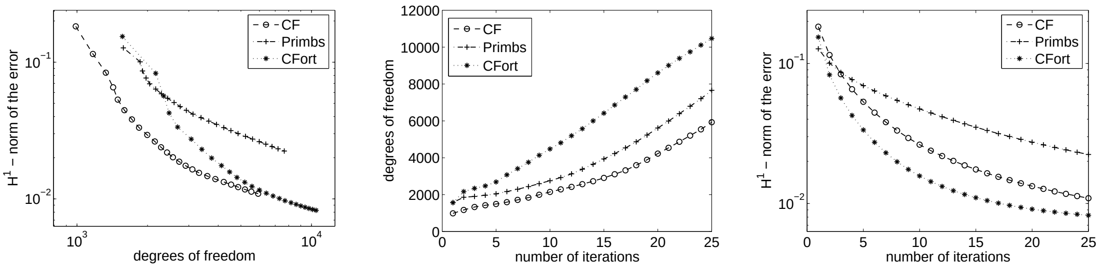

Fig. 10 Convergence history for 2d example, comparison of our wavelet bases with and without orthogonalization and wavelet bases from [24].

Acknowledgements The research of the first author has been supported by the Ministry of Education, Youth and Sports of the Czech Republic through the Research Centers LC06024. The second author has been supported by the research center 1M06047 of the Ministry of Education, Youth and Sports of the Czech Republic and by the grant 201/09/P641 of the Grant Agency of the Czech Republic.

References

1. Andersson, L.; Hall, N.; Jawerth, B.; Peters, G.: *Wavelets on the Closed Subsets of the Real Line,* in: Topics in the Theory and Applications of Wavelets, L.L. Schumaker and G. Webb (eds.), Academic Press, Boston, 1994, pp. 1–64.
2. Barsch, T.: *Adaptive Multiskalenverfahren für Elliptische Partielle Differentialgleichungen - Realisierung, Umsetzung und Numerische Ergebnisse*, Ph.D. thesis, RWTH Aachen, 2001.
3. Bittner, K.: *A New View on Biorthogonal Spline Wavelets,* preprint, Universität Ulm, 2005.
4. Bittner, K.: *Biorthogonal Spline Wavelets on the Interval,* in: Wavelets and Splines, Athens, 2005, Mod. Methods Math., Nashboro Press, Brentwood, TN, 2006, pp. 93-104.
5. Burstedde, C.: *Fast Optimized Wavelet Methods for Control Problems Constrained by Elliptic PDEs,* Ph.D. thesis, Universität, Bonn, 2005.
6. Carnicer, J.M.; Dahmen, W.; Peňa, J.M.: *Local decomposition of refinable spaces,* Appl. Comp. Harm. Anal. **6**, (1999), pp. 1-52.
7. Canuto, C.; Tabacco, A.; Urban, K.: *The Wavelet Element Method, Part I: Construction and Analysis,* Appl. Comp. Harm. Anal. **6**, (1999), pp. 1-52.
8. Chui, C.K.; Quak, E.:, *Wavelets on a Bounded Interval,* in: Numerical Methods of Approximation Theory (Braess, D.; Schumaker, L.L., eds.), Birkhäuser, 1992, pp. 53-75.
9. Cohen, A.; Daubechies, I.; Feauveau, J.-C.: *Biorthogonal Bases of Compactly Supported Wavelets*. Comm. Pure and Appl. Math. **45**, (1992), pp. 485-560.
10. Cohen, A.; Daubechies, I.; Vial, P.: *Wavelets on the Interval and Fast Wavelet Transforms,* Appl. Comp. Harm. Anal. **1**, (1993), pp. 54-81.
11. Černá, D.: *Biorthogonal Wavelets*, Ph.D. thesis, Charles University, Prague, 2008.
12. Dahlke, S.; Fornasier, M.; Primbs, M.; Raasch, T.; Werner, M.: *Nonlinear and Adaptive Frame Approximation Schemes for Elliptic PDEs: Theory and Numerical Experiments,* preprint, Philipps-Universität Marburg, 2007.
13. Dahmen, W.: *Stability of Multiscale Transformations,* J. Fourier Anal. Appl. **4**, (1996), pp. 341-362.
14. Dahmen, W.: *Multiscale Analysis, Approximation, and Interpolation Spaces,* in: Approximation Theory VIII, (Chui, C.K.; Schumaker, L.L, eds.), World Scientific Publishing Co., 1995, pp. 47-88.
15. Dahmen, W.; Han, B.; Jia R.Q.; Kunoth A.: *Biorthogonal Multiwavelets on the Interval: Cubic Hermite Splines,* Constr. Approx. **16**, (2000), no. 2, pp. 221-259.

16. Dahmen, W.; Kunoth, A.; Urban, K.: *Biorthogonal Spline Wavelets on the Interval - Stability and Moment Conditions,* Appl. Comp. Harm. Anal. **6**, (1999), pp. 132-196.
17. Dahmen, W.; Kunoth, A.; Urban, K.: *Wavelets in Numerical Analysis and Their Quantitative Properties,* in: Surface fitting and multiresolution methods **2** (Le Méhauté, A.; Rabut, C.; Schumaker, L., ed.), 1997, pp. 93–130.
18. Dahmen, W.; Miccheli, C.A.: *Banded Matrices with Banded Inverses, II: Locally Finite Decomposition of Spline Spaces,* Constr. Appr. **9**, (1993), pp. 263-281.
19. Jia, R.Q.: *Stable Bases of Spline Wavelets on the Interval*, in: Wavelets and Splines, Athens 2005, Mod. Methods Math., Nashboro Press, Brentwood, TN, 2006, pp. 120-135.
20. Jia, R.Q.: *Spline Wavelets on the Interval with Homogeneous Boundary Conditions,* Adv. Comput. Math. **30**, (2009), no. 2, pp. 177–200.
21. Jia, R.Q.; Jiang, Q.T.: *Spectral Analysis of the Transition Operator and its Applications to Smoothness Analysis of Wavelets,* SIAM J. Matrix Anal. Appl. **24**, (2003), no. 4, pp. 1071–1109.
22. Jia, R.Q.; Liu, S.T.: *Wavelet Bases of Hermite Cubic Splines on the Interval,* Adv. Comput. Math. **25**, (2006), no. 1–3, pp. 23–39.
23. Pabel, R.: *Wavelet Methods for PDE Constrained Elliptic Control Problems with Dirichlet Boundary Control*, thesis, Universität Bonn, 2005.
24. Primbs, M.: *Stabile biorthogonale Spline-Waveletbasen auf dem Intervall,* dissertation, Universität Duisburg-Essen, 2006.
25. Primbs, M.: *New Stable Biorthogonal Spline Wavelets on the Interval,* preprint, Universität Duisburg-Essen, 2007.
26. Primbs, M.: *Technical Report for the Paper: 'New Stable Biorthogonal Spline Wavelets on the Interval'*, Universität Duisburg-Essen, 2007.
27. Raasch, T.: *Adaptive Wavelet and Frame Schemes for Elliptic and Parabolic Equations,* Ph.D. thesis, Marburg, 2007.
28. Schneider, A.: *Biorthogonal Cubic Hermite Spline Multiwavelets on the Interval with Complementary Boundary Conditions,* preprint, Philipps-Universität Marburg, 2007.
29. Schumaker, L.L.: *Spline Functions: Basic Theory*, Wiley-Interscience, 1981.
30. Stevenson, R.: *Adaptive Solution of Operator Equations Using Wavelet Frames,* SIAM J. Numer. Anal. **41**, (2003), pp. 1074-1100.
31. Stevenson, R.: *On the Compressibility of Operators in Wavelet Coordinates,* SIAM J. Math. Anal. **35**, (2004), no. 5, pp. 1110-1132.
32. Stevenson, R.: *Composite Wavelet Bases with Extended Stability and Cancellation Properties,* SIAM J. Numer. Anal. **45**, (2007), no. 1, pp. 133-162.
33. Grivet Talocia, S.; Tabacco, A.: *Wavelets on the Interval with Optimal Localization,* in: Math. Models Meth. Appl. Sci. **10**, 2000, pp. 441-462.
34. Turcajova, R.: *Numerical Condition of Discrete Wavelet Transforms.* SIAM J. Matrix Anal. Appl. **18**, (1997), pp. 981-999.
35. Urban, K.: *Wavelet Methods for Elliptic Partial Differential Equations.* Oxford University Press, 2009.
36. Weyrich, N.: *Spline Wavelets on an Interval*, in: Wavelets and Allied Topics, 2001, pp. 117-189.