Matematika 1A (Fakulta strojní)

Cvičení z matematické analýzy

TU v Liberci

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Kapitola 1

Množinové operace, číselné obory a výrokový počet

Příklad 1.1. Určete všechny podmnožiny množiny M:

a) $M = \{-4, 3, 5\}$ []
b) $M = \{0, 1, 2\}$ []

Příklad 1.2. Rozhodněte o rovnosti množin:

a) $M = \emptyset$, $N = \{\emptyset\}$ []
b) $M = \{-2, -1, 0\}$, $N = \{x \in \mathbb{Z} : |x + 1| \leq 1\}$ []
c) $M = \{-4, -3, -2, -1, 0\}$, $N = \{x \in \mathbb{Z} : |x + 2| < 2\}$ []
d) $M = \{-2, -1, 0\}$, $N = \{x \in \mathbb{Z} : |x + 1| \leq 1\}$ []
e*) $M = \{-2, -1, 0\}$, $N = \{x \in \mathbb{Z} : |x + 2| < 2\}$ []

Příklad 1.3. Zakreslete následující množiny a určete jejich sjednocení, průnik, rozdíl a doplněk v množině D:

a) $M = \{1, 2, 3\}$, $N = \{x \in \mathbb{Z} : |x - 2| \leq 2\}$, $D = \mathbb{Z}$ []
b) $M = (-4, -3)$, $N = \langle -2, 2\rangle$, $D = \mathbb{R}$ []
c*) $M = \langle 1, 2)$, $N = \{x \in \mathbb{R} : |x - 3| < 1\}$, $D = \mathbb{R}_0^+$ []
d) $M = \{2, 3, 5, 11, \ldots\}$ (mn. všech prvočísel),
$N = \{2, 4, 6, 8, \ldots\}$ (mn. všech sudých císel), $D = \mathbb{Z}$ []
e) $M = \mathbb{R}_0^-$, $N = \{x \in \mathbb{R} : |x - 2| - 2x + 3 < 0\}$, $D = \mathbb{R}$ []
f*) $M = (4, 8)$, $N = \{x \in \mathbb{R} : |x - 6| \leq 2\}$, $D = \mathbb{R}^+$ []
g) $M = \{x \in \mathbb{Z} : |x| \leq 4\}$, $N = \{x \in \mathbb{Z} : x^2 < 25\}$, $D = \mathbb{Z}$ []
h) $M = \mathbb{Q}^+$, $N = \mathbb{I}^+$, $D = \mathbb{R}^+$ []
i) $M = \mathbb{R}$, $N = \mathbb{R}^*$, $D = \mathbb{R}^*$ []

Příklad 1.4. Pro množiny $A = (-\infty, 2)$, $B = \langle 1, 3\rangle$ a $C = \langle -1, 1\rangle \cup \langle 2, +\infty)$ určete:

a) $(A \cup B) \cap C$ []
b) $(C \cup B) \cap A$ []
c) $A \cup B \cup C$ []
d) $A \cap B \cap C$ []

Příklad 1.5. Nechť A, B, C jsou množiny. Ověřte platnost de Morganových pravidel:

a) $A - (B \cap C) = (A - B) \cup (A - C)$,
b) $A - (B \cup C) = (A - B) \cap (A - C)$

Příklad 1.6. Určete minimum, maximum, infimum a supremum následujících množin v $I\!R^*$:

a) $M = (0,1)$ []
b) $M = \langle 0,1\rangle$ []
c) $M = \langle 1,2)$ []
d) $M = \langle -1,+\infty)$ []
e) $M = (-\infty,1)$ []
f) $M = \emptyset$ []
g) $M = \langle 0,1\rangle \cup \langle 2,3\rangle$ []

Příklad 1.7. Utvořte negaci výroků, je-li:

a*) $A : \forall x \in I\!R : x^2 + 1 > 0$ [$\neg A : \exists x \in I\!R : x^2 + 1 \leq 0$]
b*) A : Všichni lidé jsou menší než 280 cm. [$\neg A$: Existuje alespoň jeden člověk větší nebo roven 280 cm.]

Příklad 1.8. Nechť A a B je dvojice následujících výroků:

a*) $A : x \in I\!R : x < 2$, $B : x \in I\!R : x \geq -1$ [$p(A \wedge B) = 1$ pro $x \in \langle -1,2)$, $p(A \vee B) = 1$ pro $x \in I\!R$]

Pro která $x \in I\!R$ je konjunkce $A \wedge B$ pravdivá a pro která $x \in I\!R$ je disjunkce $A \vee B$ pravdivá?

Příklad 1.9. Utvořte disjunkci dvou výroků p a q a určete pravdivost složeného výroku, je-li:

a*) $p : 2|12$, $q : 5|12$ [$p(p \vee q) = 1$]
b*) $p : \forall n \in \mathbb{N} : n^2 + n + 11$ je prvočíslo.
$q : \forall n \in \mathbb{N} : n^2 + n + 11$ je liché. [$p(p \vee q) = 1$]

Příklad 1.10. Určete pravdivost níže uvedených složených výroků pro $A : \sqrt{4} = -2$ a $B : 3 + 4 = 7$:

a) $A \Rightarrow B$ []
b) $\neg A \Rightarrow B$ []
c) $\neg B \Rightarrow A$ []
d) $\neg B \Rightarrow A$ []

Příklad 1.11. Určete pravdivost níže uvedených složených výroků pro $p : e = 6$, $q : \pi^2 = 16$ a $r : 1 < 2$:

a) $(p \vee q) \wedge r$ []
b) $(p \wedge q) \vee r$ []

Příklad 1.12. Utvořte pravdivostní tabulku hodnot složeného výroku a rozhodněte o platnosti ekvivalence:

a*) $(X \Rightarrow Y) \Leftrightarrow (\neg X \vee Y)$ [tautologie]
b) $\neg(\neg X) \Leftrightarrow X$ []
c) $\neg(X \wedge Y) \Leftrightarrow (\neg X \vee \neg Y)$ []
d) $\neg(X \vee Y) \Leftrightarrow (\neg X \wedge \neg Y)$ []
e*) $\neg(X \Rightarrow Y) \Leftrightarrow (X \wedge \neg Y)$ [tautologie]
f) $\neg(X \Leftrightarrow Y) \Leftrightarrow [(X \wedge \neq Y) \vee (\neg X \wedge Y)]$ []

Kapitola 2

Posloupnosti

Příklad 2.1. Určete, zda jsou následující posloupnosti ohraničené a rozhodněte o jejich monotonii:

a*) $a_n = 3\cos(n^2)$ []

b) $a_n = 2^{1-n}$ [kles., omez.]

c) $a_n = 1 + \dfrac{(-1)^n}{n}$ []

d) $a_n = \dfrac{n}{2^n}$ [nerost., omez.]

e) $a_n = (-1)^n n^2$ [osciluje, neomez.]

f) $a_n = n^3 - n^2$ [rost., zdola omez.]

g) $a_n = \dfrac{2^n}{n!}$ []

h) $a_n = \dfrac{1}{n^2}$ [kles., omez.]

i) $a_n = \sqrt[n]{2}$ [kles., omez.]

j) $a_n = \sqrt{n}$ [rost., zdola omez.]

Příklad 2.2. Vypočtěte:

a*) $\displaystyle\lim_{n\to\infty} \frac{-2n^3 - 5n + 7}{5n^2 + n - 8}$ $[-\infty]$

b*) $\displaystyle\lim_{n\to\infty} \frac{5n^2 + 8n + 1}{7n^2 + 8n - 1}$ $\left[\frac{5}{7}\right]$

c*) $\displaystyle\lim_{n\to\infty} \frac{3n + 3}{n^3 - 1}$ $[0]$

d) $\displaystyle\lim_{n\to\infty} \frac{6n^2 + 5n - 2}{1 - 2n + 6n^2}$ $[1]$

e) $\displaystyle\lim_{n\to\infty} \frac{2n^2 + 1}{n^2 - 3n + 1}$ $[2]$

f) $\displaystyle\lim_{n\to\infty} \frac{n^2 + 2n + 5}{n^3 + 1}$ $[0]$

g) $\displaystyle\lim_{n\to\infty} \frac{n^3 + 1}{n^2 + 2n + 5}$ $[+\infty]$

h*) $\displaystyle\lim_{n\to\infty} \frac{n^2 - 1}{(n + 1)^3}$ $[0]$

i) $\displaystyle\lim_{n\to\infty} \frac{(n + 2)^5(2n - 1)}{(n - 1)^6}$ $[2]$

j) $\displaystyle\lim_{n\to\infty} (n^2 + 5n - 1)$ $[+\infty]$

k) $\displaystyle\lim_{n\to\infty} (n^2 - 5n - 1)$ $[+\infty]$

l) $\displaystyle\lim_{n\to\infty} (-n^2 + 5n)$ $[-\infty]$

m) $\displaystyle\lim_{n\to\infty} \frac{-5n^2 + 8n + 4}{1 + 2n + 3n^2}$ $\left[-\frac{5}{3}\right]$

n) $\displaystyle\lim_{n\to\infty} \frac{-8n^2 + 6n + 7}{2n + 5}$ $[-\infty]$

o) $\displaystyle\lim_{n\to\infty} \frac{-8n + 3}{9 - 2n - 4n^2}$ $[0]$

p) $\displaystyle\lim_{n\to\infty} \frac{(2n + 1)(3n + 2)(3n - 8)}{n^3 - n^2 + 1}$ $[18]$

q) $\displaystyle\lim_{n\to\infty} \frac{(n + 1)^4 - (n - 1)^4}{(n + 1)^3 - (n - 1)^3}$ $[+\infty]$

r*) $\displaystyle\lim_{n\to\infty} \left(\frac{2n^5 + 3n - 2}{n^5 - 3n^2 + 1}\right)^2$ $[4]$

Příklad 2.3. Vypočtěte:

a) $\lim\limits_{n\to\infty} \left(\sqrt{9n^2-4}-2n\right)$ $[+\infty]$

b) $\lim\limits_{n\to\infty} \left(\sqrt{9n^2-4}-3n\right)$ $[0]$

c) $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n^2+n}-\sqrt{n^2+2}}$ $[2]$

d*) $\lim\limits_{n\to\infty} \left(\sqrt{n+2}-\sqrt{n-2}\right)$ $[0]$

e) $\lim\limits_{n\to\infty} \left(\sqrt{n+\sqrt{n}}-\sqrt{2n+1}\right)$ $[-\infty]$

f) $\lim\limits_{n\to\infty} \left(\sqrt{4n^2-5}-2n\right)$ $[0]$

g) $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n+1}-\sqrt{n-1}}$ $[+\infty]$

h) $\lim\limits_{n\to\infty} \left(\sqrt{4n^2-n}-n\right)$ $[+\infty]$

i*) $\lim\limits_{n\to\infty} \left(\sqrt{4n^2-n}-2n\right)$ $\left[-\dfrac{1}{4}\right]$

j) $\lim\limits_{n\to\infty} \dfrac{-3n+\sqrt{2n+1}}{n-\sqrt{n}}$ $[-3]$

k) $\lim\limits_{n\to\infty} \left(\sqrt{n-1}-\sqrt{n}\right)$ $[0]$

l) $\lim\limits_{n\to\infty} \left(\sqrt{5n^2+3n}-\sqrt{5n^2+2n}\right)$ $\left[\dfrac{1}{2\sqrt{5}}\right]$

m) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^2+1}-\sqrt{n}}{n}$ $[1]$

n) $\lim\limits_{n\to\infty} \left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$ $\left[\dfrac{1}{2}\right]$

o) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^2+1}}{n+1}$ $[1]$

p) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^3-1}}{n+1}$ $[+\infty]$

Příklad 2.4. Vypočtěte:

a) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^{3n}$ $[e^3]$

b) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^{n+5}$ $[e]$

c) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^{7n+4}$ $[e^7]$

d) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{5n}\right)^{n}$ $[\sqrt[5]{e}]$

e) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{4n}\right)^{5n+8}$ $[\sqrt[4]{e^5}]$

f) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{2n+3}\right)^{7n+6}$ $[\sqrt{e^7}]$

g) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^{6n}$ $[e^6]$

h) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^{3n+2}$ $[e^3]$

i) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{3n}\right)^{n}$ $[e^{\frac{1}{3}}]$

j*) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{2n}\right)^{3n+6}$ $[e^{\frac{3}{2}}]$

k) $\lim\limits_{n\to\infty} \left(\dfrac{3n+2}{3n+1}\right)^{n}$ $[e^{\frac{1}{3}}]$

l) $\lim\limits_{n\to\infty} \left(\dfrac{n+1}{n}\right)^{k}$, $k\in\mathbb{Z}$ $[1]$

m) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{2n}\right)^{n}$ $[e^{\frac{1}{2}}]$

n) $\lim\limits_{n\to\infty} \left(1-\dfrac{1}{n}\right)^{n}$ $[e^{-1}]$

o) $\lim\limits_{n\to\infty} \left(1-\dfrac{1}{n^2}\right)^{n^2-1}$ $[e^{-1}]$

p*) $\lim\limits_{n\to\infty} \left(1-\dfrac{4}{n}\right)^{n}$ $[e^{-4}]$

q*) $\lim\limits_{n\to\infty} \left(\dfrac{2n+3}{2n-1}\right)^{3n+2}$ $[e^6]$

r) $\lim\limits_{n\to\infty} \left(1+\dfrac{2}{2n+1}\right)^{1-n}$ $[e^{-1}]$

s*) $\lim\limits_{n\to\infty} \left(\dfrac{3n}{3n-1}\right)^{3n}$ $[e]$

t) $\lim\limits_{n\to\infty} \left(\dfrac{2n}{n-1}\right)^{2n}$ $[+\infty]$

u) $\lim\limits_{n\to\infty} \left(\dfrac{2n}{3n-1}\right)^{n}$ $[0]$

v) $\lim\limits_{n\to\infty} \left(\dfrac{3n+1}{2n-1}\right)^{4n-1}$ $[+\infty]$

v) $\lim\limits_{n\to\infty} \left(\dfrac{4-n}{n+2}\right)\left(\dfrac{n+2}{n+1}\right)^{2-n}$ $[-e^{-1}]$

w) $\lim\limits_{n\to\infty} \left(\dfrac{2n+5}{5-n}\right)\left(\dfrac{2n-3}{2n+5}\right)^{2n+5}$ $[-2e^{-8}]$

Příklad 2.5. Vypočtěte:

a*) $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n^3+4n+5}$ [0]

b) $\lim\limits_{n\to\infty} \left(\dfrac{1}{n}\cos\left(\dfrac{n^2+1}{2n-1}\right)\right)$ [0]

c) $\lim\limits_{n\to\infty} \dfrac{\sin(n^2+1)}{n}$ [0]

d) $\lim\limits_{n\to\infty} \dfrac{n\cdot\sin n^2}{n^2+1}$ [0]

e*) $\lim\limits_{n\to\infty} \dfrac{\cos n}{n+1}$ [0]

f) $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n^2-5n+1}$ [0]

g) $\lim\limits_{n\to\infty} \dfrac{\cos\left(e^{n^2+n+1}\right)}{n}$ [0]

h) $\lim\limits_{n\to\infty} \dfrac{n\sin n!}{n^2+1}$ [0]

i) $\lim\limits_{n\to\infty} \dfrac{\sin(n\pi/2)}{n}$ [0]

j) $\lim\limits_{n\to\infty} \dfrac{5\cos n}{3n+7}$ [0]

k) $\lim\limits_{n\to\infty} \dfrac{\sin 5n}{n^2}$ [0]

l) $\lim\limits_{n\to\infty} \dfrac{n^2+\sin n}{n}$ [$+\infty$]

Příklad 2.6. Vypočtěte:

a*) $\lim\limits_{n\to\infty} \sqrt[2n]{n}$ [1]

b) $\lim\limits_{n\to\infty} \sqrt[n]{n^7}$ [1]

c) $\lim\limits_{n\to\infty} \sqrt[2n]{3^n}$ [$\sqrt{3}$]

d*) $\lim\limits_{n\to\infty} \sqrt[n]{2^n+3^{n+1}}$ [3]

e) $\lim\limits_{n\to\infty} \sqrt[5n]{n}$ [1]

f*) $\lim\limits_{n\to\infty} \sqrt[7n]{6^n}$ [$\sqrt[7]{6}$]

g*) $\lim\limits_{n\to\infty} \sqrt[2n]{9^n+4^n}$ [3]

h*) $\lim\limits_{n\to\infty} \sqrt[n]{n+5}$ [1]

i) $\lim\limits_{n\to\infty} \sqrt[n]{n^2+1}$ [1]

Příklad 2.7. Vypočtěte:

a*) $\lim\limits_{n\to\infty} \dfrac{(-1)^n+2}{(-1)^n-2}$ [neex.]

b) $\lim\limits_{n\to\infty} \dfrac{2^n+(-2)^n}{2\cdot 4^n}$ [0]

c) $\lim\limits_{n\to\infty} \dfrac{(-1)^n+2}{3^n(2-(-1)^n)}$ [0]

d) $\lim\limits_{n\to\infty} \dfrac{2^n}{(\frac{1}{2})^n+1}$ [$+\infty$]

e) $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{2}-1}{\sqrt[n]{2}-2}$ [0]

f) $\lim\limits_{n\to\infty} \dfrac{2^n-1}{2^n+2}$ [1]

g) $\lim\limits_{n\to\infty} \left(\dfrac{3^n}{1-3^n}+\sqrt[n]{3}\right)$ [0]

h*) $\lim\limits_{n\to\infty} \dfrac{6^n+4^n-2^n}{6^{n+1}-3^n}$ $\left[\dfrac{1}{6}\right]$

i) $\lim\limits_{n\to\infty} \dfrac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}$ $\left[\dfrac{1}{3}\right]$

j) $\lim\limits_{n\to\infty} (-1)^{n^2+2n+1}$ [neex.]

Příklad 2.8. Vypočtěte:

a) $\lim\limits_{n\to\infty} \dfrac{(n+2)!-3n!}{(n+2)!+1}$ [1]

b) $\lim\limits_{n\to\infty} \dfrac{3(n+2)!-(n+1)!}{(n+3)!}$ [0]

c) $\lim\limits_{n\to\infty} \dfrac{(n+1)!-2n!}{3(n+1)!+1}$ $\left[\dfrac{1}{3}\right]$

d) $\lim\limits_{n\to\infty} \dfrac{(2n+1)!+(2n-1)!}{(2n+1)!-(2n)!}$ [1]

Příklad 2.9. Vypočtěte:

a) $\lim\limits_{n\to\infty} \dfrac{1+2+3+\cdots+n}{n^2}$ $\left[\dfrac{1}{2}\right]$

b) $\lim\limits_{n\to\infty} \left(\dfrac{1+2+3+\cdots+n}{n+2}-\dfrac{n}{2}\right)$ $\left[-\dfrac{1}{2}\right]$

c) $\lim\limits_{n\to\infty} \left(\dfrac{1}{n^2}+\dfrac{2}{n^2}+\cdots+\dfrac{n-1}{n^2}\right)$ $\left[\dfrac{1}{2}\right]$

d) $\lim\limits_{n\to\infty} \dfrac{1+\frac{1}{3}+\frac{1}{9}+\cdots+\frac{1}{3^n}}{1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}}$ $\left[\dfrac{3}{4}\right]$

e) $\lim\limits_{n\to\infty} \dfrac{1+\frac{1}{3}+\frac{1}{9}+\cdots+\frac{1}{3^n}}{1+\frac{1}{4}+\frac{1}{16}+\cdots+\frac{1}{4^n}}$ $\left[\dfrac{9}{8}\right]$

f) $\lim\limits_{n\to\infty} \dfrac{1+2+3+\cdots+n}{1+2+3+\cdots+(n-1)}$ [1]

Kapitola 3

Reálné funkce reálné proměnné

Příklad 3.1. Určete definiční obory funkcí:

a) $f: y = \log_{\frac{1}{3}} \left(\dfrac{x-3}{x+3}\right)$ $\quad [D_f = (-\infty; -3) \cup (3; +\infty)]$

b) $f: y = \log_{\frac{1}{3}}(x-3) - \log_{\frac{1}{3}}(x+3)$ $\quad [D_f = (3; +\infty)]$

c) $f: y = \sqrt{\ln(x^2-1)}$ $\quad \left[D_f = \left(-\infty; -\sqrt{2}\right\rangle \cup \left\langle \sqrt{2}; +\infty\right)\right]$

d) $f: y = \dfrac{\sqrt{x^2-1}}{\ln(2x-3)}$ $\quad \left[D_f = \left(\dfrac{3}{2}; 2\right) \cup (2; +\infty)\right]$

e) $f: y = \ln(2-x)$ $\quad [D_f = (-\infty; 2)]$

f) $f: y = \ln(x^2-4)$ $\quad [D_f = (-\infty; -2) \cup (2; +\infty)]$

g) $f: y = e^{\sqrt{x^2-1}}$ $\quad [D_f = (-\infty; -1\rangle \cup \langle 1; +\infty)]$

h) $f: y = \ln(e^x - e^{-x})$ $\quad [D_f = (0; +\infty)]$

i) $f: y = \ln\left|\sqrt{1-|x|}\right| + \dfrac{1}{\sqrt{x+\frac{1}{2}}}$ $\quad \left[D_f = \left(-\dfrac{1}{2}; 1\right)\right]$

j) $f: y = \log_a(x-4)(x-1)$ $\quad [D_f = (-\infty; 1) \cup (4; +\infty),\ a \in (0; 1) \cup (1, +\infty)]$

k) $f: y = \ln \dfrac{e^x - 1}{e^x}$ $\quad [D_f = (0; +\infty)]$

l) $f: y = \ln e^x + \ln \sqrt{3 - 2x - x^2}$ $\quad [D_f = (-3; 1)]$

m) $f: y = \log_a(1-x^2)$ $\quad [D_f = (-1; 1),\ a \in (0; 1) \cup (1; +\infty)]$

n) $f: y = \log_a \sqrt{x+2}$ $\quad [D_f = (-2; +\infty),\ a \in (0; 1) \cup (1; +\infty)]$

o) $f: y = \log_a \dfrac{x-3}{x+2}$ $\quad [D_f = (-\infty; -2) \cup (3; +\infty),\ a \in (0; 1) \cup (1; +\infty)]$

p) $f: y = \dfrac{1}{x^2-9} - \ln(x^3 - x)$ $\quad [D_f = (-1; 0) \cup (1; 3) \cup (3; +\infty)]$

q) $f: y = 1 - \ln(x^2 - 5x + 6)$ $\quad [D_f = (-\infty; 2) \cup (3; +\infty)]$

r) $f: y = \ln(-x^2 + x + 2)$ $\quad [D_f = (-1; 2)]$

s) $f: y = \ln \dfrac{x-1}{x+3} + \sqrt{x^2-4}$ $\quad [D_f = (-\infty; -3) \cup \langle 2; +\infty)]$

t) $f: y = \sqrt{\dfrac{x^2 - 5x + 4}{x+2}} + 2 \ln \dfrac{x^2+1}{x}$ $\quad [D_f = (0; 1\rangle \cup \langle 4; +\infty)]$

u) $f: y = \ln|\ln(-\ln x)|$ $\quad [D_f = (0; 1)]$

v) $f: y = \ln(2 - |2x^2 + 10x + 12|)$ $\quad \left[D_f = \left(\dfrac{-5-\sqrt{5}}{2}; \dfrac{-5+\sqrt{5}}{2}\right)\right]$

Příklad 3.2. Určete definiční obory funkcí:

a) $f : y = \dfrac{1}{x}$ $[D_f = I\!R - \{0\}]$

b) $f : y = \dfrac{x-1}{x^2-4}$ $[D_f = I\!R - \{-2; 2\}]$

c) $f : y = \sqrt{x-1}$ $[D_f = \langle 1; +\infty)]$

d) $f : y = \sqrt{x^2-3x+2}$ $[D_f = (-\infty; 1\rangle \cup \langle 2; +\infty)]$

e) $f : y = \sqrt[4]{\dfrac{x+1}{x-1}}$ $[D_f = (-\infty; -1\rangle \cup (1; +\infty)]$

f) $f : y = \dfrac{1}{x+1} + \sqrt[6]{x+3}$ $[D_f = \langle -3; -1) \cup (-1; +\infty)]$

g) $f : y = \sqrt{x+4} + \sqrt[4]{x^2-5x+6}$ $[D_f = \langle -4; 2\rangle \cup \langle 3; +\infty)]$

h) $f : y = x^2 + \sqrt[3]{x}$ $[D_f = I\!R]$

i) $f : y = \sqrt{2+x-x^2}$ $[D_f = \langle -1, 2\rangle]$

j) $f : y = \sqrt{\dfrac{x}{3-x^2}}$ $[D_f = (-\infty; -\sqrt{3}) \cup \langle 0; \sqrt{3})]$

k*) $f : y = \sqrt{x^2+x-6}$ $[D_f = (-\infty; -3\rangle \cup \langle 2; +\infty)]$

l) $f : y = \dfrac{4-x^2}{x^2-2x-3}$ $[D_f = I\!R - \{-1; 3\}]$

m) $f : y = \dfrac{\sqrt{3-|x|}}{||x|-2|}$ $[D_f = \langle -3; -2) \cup (-2; 2) \cup (2; 3\rangle]$

n) $f : y = \dfrac{1}{1+\sqrt[3]{x}}$ $[D_f = (-\infty; -1) \cup (-1; +\infty)]$

o) $f : y = \dfrac{-1}{1-\sqrt{x}}$ $[D_f = \langle 0; 1) \cup (1; +\infty)]$

p) $f : y = \dfrac{1}{x-\sqrt{x}}$ $[D_f = (0; 1) \cup (1; +\infty)]$

Příklad 3.3. Určete definiční obory funkcí:

a) $f: y = \sqrt{\ln(\cos x)}$ $\left[D_f = \bigcup_{k\in\mathbb{Z}} \{0 + 2k\pi\}\right]$

b) $f: y = \arcsin \frac{x-3}{2} + \ln(x^3 - x)$ $[D_f = (1; 5\rangle]$

c) $f: y = \sqrt{\ln\left(\frac{5x - x^2}{4}\right)} + \arcsin \frac{3}{x}$ $[D_f = \langle 3; 4\rangle]$

d) $f: y = \operatorname{tg}(x + \pi)$ $\left[D_f = \bigcup_{k\in\mathbb{Z}} \left(\frac{\pi}{2} + k\pi; \frac{3\pi}{2} + k\pi\right)\right]$

e) $f: y = \frac{1}{\cos x}$ $\left[D_f = I\!R - \bigcup_{k\in\mathbb{Z}} \left\{\frac{\pi}{2} + k\pi\right\}\right]$

f) $f: y = \frac{1}{\sin x}$ $\left[D_f = I\!R - \bigcup_{k\in\mathbb{Z}} \{k\pi\}\right]$

g) $f: y = \ln(\sin x)$ $\left[D_f = \bigcup_{k\in\mathbb{Z}} (0 + 2k\pi; \pi + 2k\pi)\right]$

h) $f: y = \sin(\ln x)$ $[D_f = (0; +\infty)]$

i) $f: y = \arcsin(x^2 - 1)$ $[D_f = \langle -\sqrt{2}; \sqrt{2}\rangle]$

j) $f: y = \arccos(\ln x)$ $\left[D_f = \left\langle e^{-1}; e\right\rangle\right]$

k) $f: y = \frac{1}{1 - \cos x}$ $\left[D_f = I\!R - \bigcup_{k\in\mathbb{Z}} \{0 + 2k\pi\}\right]$

l) $f: y = \operatorname{tg}(2x)$ $\left[D_f = \bigcup_{k\in\mathbb{Z}} \left(-\frac{\pi}{4} + k\frac{\pi}{2}; \frac{\pi}{4} + k\frac{\pi}{2}\right)\right]$

m) $f: y = 1 - \operatorname{cotg}\left(\frac{x}{2}\right)$ $\left[D_f = I\!R - \bigcup_{k\in\mathbb{Z}} \{0 + 2k\pi\}\right]$

n) $f: y = 1 + \frac{1}{\cos x}$ $\left[D_f = I\!R - \bigcup_{k\in\mathbb{Z}} \left\{\frac{\pi}{2} + k\pi\right\}\right]$

o) $f: y = \frac{1}{1 - \sin x}$ $\left[D_f = I\!R - \bigcup_{k\in\mathbb{Z}} \left\{\frac{\pi}{2} + 2k\pi\right\}\right]$

p) $f: y = \operatorname{cotg}(2x)$ $\left[D_f = \bigcup_{k\in\mathbb{Z}} \left(0 + k\frac{\pi}{2}; \frac{\pi}{2} + k\frac{\pi}{2}\right)\right]$

q) $f: y = \sqrt{\cos x}$ $\left[D_f = \bigcup_{k\in\mathbb{Z}} \left\langle -\frac{\pi}{2} + 2k\pi; \frac{\pi}{2} + 2k\pi\right\rangle\right]$

r) $f: y = \arcsin \sqrt{\frac{1-x}{1+x}}$ $[D_f = \langle 0; 1\rangle]$

s) $f: y = \log(\cos x)$ $\left[D_f = \bigcup_{k\in\mathbb{Z}} \left(-\frac{\pi}{2} + 2k\pi; \frac{\pi}{2} + 2k\pi\right)\right]$

t) $f: y = e^{\sqrt{\frac{1}{2} - \sin x}}$ $\left[D_f = \bigcup_{k\in\mathbb{Z}} \left\langle \frac{5\pi}{6} + 2k\pi; \frac{13\pi}{6} + 2k\pi\right\rangle\right]$

u) $f: y = \sqrt{\operatorname{cotg}\left(x - \frac{\pi}{4}\right)}$ $\left[D_f = \bigcup_{k\in\mathbb{Z}} \left(\frac{\pi}{4} + 2k\pi; \frac{3\pi}{4} + 2k\pi\right\rangle\right]$

v) $f: y = \sqrt{1 - \operatorname{cotg}^2 x}$ $\left[D_f = \bigcup_{k\in\mathbb{Z}} \left\langle \frac{\pi}{4} + 2k\pi; \frac{3\pi}{4} + 2k\pi\right\rangle\right]$

Příklad 3.4. Určete definiční obory funkcí:

a) $f: y = \arcsin(2-3x)$ $\left[D_f = \left\langle \frac{1}{3}, 1 \right\rangle\right]$

b) $f: y = \arccos \frac{1-2x}{4}$ $\left[D_f = \left\langle -\frac{3}{2}, \frac{5}{2} \right\rangle\right]$

c) $f: y = \arcsin \frac{1}{2x-1}$ $\left[D_f = \left\langle 0, \frac{1}{2} \right) \cup \left(\frac{1}{2}, 1 \right\rangle\right]$

d) $f: y = \sqrt{3-x} + \arcsin\left(\frac{3-2x}{5}\right)$ $[D_f = \langle -1, 3 \rangle]$

e) $f: y = \arcsin(1-x) + \ln(\ln x)$ $[D_f = (1, 2\rangle]$

f) $f: y = \sqrt{\operatorname{arctg}\left(\frac{x+3}{2}\right) - \frac{\pi}{4}}$ $[D_f = \langle -1, +\infty)]$

g) $f: y = \frac{\ln(2x-3)}{\sqrt{x^2-1}} + \arcsin \frac{x-4}{7}$ $\left[D_f = \left(\frac{3}{2}, 11 \right\rangle\right]$

h) $f: y = \arcsin \frac{2x}{1+x}$ $\left[D_f = \left\langle -\frac{1}{3}, 1 \right\rangle\right]$

i) $f: y = \arcsin(x) + x\sqrt{1-x^2}$ $[D_f = \langle -1, 1 \rangle]$

j) $f: y = \arccos(\ln x^3)$ $\left[D_f = \left\langle e^{-\frac{1}{3}}, e^{\frac{1}{3}} \right\rangle\right]$

k) $f: y = \frac{1}{1-\cos x}$ $\left[D_f = \mathbb{R} - \bigcup_{k\in\mathbb{Z}} \{2k\pi\}\right]$

l) $f: y = \operatorname{tg}(4x)$ $\left[D_f = \mathbb{R} - \bigcup_{k\in\mathbb{Z}} \left\{\frac{\pi}{8} + k\frac{\pi}{4}\right\}\right]$

m) $f: y = \left(\frac{x-2}{x+1}\right)^{\arccos\sqrt{x^2-1}}$ $\left[D_f = \left\langle -\sqrt{2}, -1 \right)\right]$

n) $f: y = \arccos \frac{3x+2}{4}$ $\left[D_f = \left\langle -2, \frac{2}{3} \right\rangle\right]$

o) $f: y = \frac{1}{\arccos(\ln x^3)}$ $\left[D_f = \left\langle e^{-\frac{1}{3}}, e^{\frac{1}{3}} \right)\right]$

p) $f: y = \arcsin\left(\frac{2-x}{2x+1}\right)$ $\left[D_f = (-\infty, -3\rangle \cup \left\langle \frac{1}{3}, +\infty \right)\right]$

Příklad 3.5. Zjistěte, zda jsou dané funkce sudé nebo liché, příp. ani sudé ani liché:

a*) $f: y = \ln\left(\dfrac{2-x}{2+x}\right)$ $\quad [D_f = (-2; 2)$, lichá$]$

b) $f: y = \dfrac{e^x+1}{e^x-1}$ $\quad [D_f = (-\infty; 0) \cup (0; +\infty)$, lichá$]$

c) $f: y = \dfrac{x}{|x|}$ $\quad [D_f = (-\infty; 0) \cup (0; +\infty)$, lichá$]$

d) $f: y = 4x^2 + 1$ $\quad [D_f = I\!R$, sudá$]$

e) $f: y = \operatorname{tg} x$ $\quad \left[D_f = I\!R - \bigcup\limits_{k\in\mathbb{Z}} \left\{\dfrac{\pi}{2} + k\pi\right\}, \text{lichá}\right]$

f) $f: y = \sin x - \cos x$ $\quad [D_f = I\!R$, ani sudá, ani lichá$]$

g) $f: y = \cosh x$ $\quad [D_f = I\!R$, sudá$]$

h) $f: y = \sinh x$ $\quad [D_f = I\!R$, lichá$]$

i) $f: y = x\dfrac{e^x+1}{e^x-1}$ $\quad [D_f = I\!R - \{0\}$, sudá$]$

j) $f: y = \dfrac{1}{2}x^3 + x$ $\quad [D_f = I\!R$, lichá$]$

k) $f: y = \dfrac{5x^3}{|x|}$ $\quad [D_f = I\!R - \{0\}$, lichá$]$

l) $f: y = x \sin x$ $\quad [D_f = I\!R$, sudá$]$

m*) $f: y = x\sin(3x) + x^2 \cos x + \dfrac{1}{x^2}$ $\quad [D_f = I\!R - \{0\}$, sudá$]$

n) $f: y = \sqrt[5]{x^2}$ $\quad [D_f = I\!R$, sudá$]$

o) $f: y = \sqrt[4]{x^3}$ $\quad [D_f = \langle 0; +\infty)$, ani sudá, ani lichá$]$

p) $f: y = \dfrac{1}{\sqrt[3]{x}}$ $\quad [D_f = I\!R - \{0\}$, lichá$]$

Příklad 3.6. Zjistěte, zda jsou dané funkce periodické a v kladném případě určete jejich nejmenší periodu:

a) $f: y = \operatorname{tg}(3x) + \sin(6x)$ $\quad \left[\dfrac{\pi}{3}\right]$

b) $f: y = \sin^2 x$ $\quad [\pi]$

c) $f: y = 1 + \cos x$ $\quad [2\pi]$

d) $f: y = \sin(2x)$ $\quad [\pi]$

e) $f: y = \cos\left(\dfrac{x}{2} + \pi\right)$ $\quad [4\pi]$

f) $f: y = 2\sin(2x - \pi)$ $\quad [\pi]$

g) $f: y = 1 + \cos(2x + \pi)$ $\quad [\pi]$

h) $f: y = \cos\left(\dfrac{x}{2}\right)$ $\quad [4\pi]$

i) $f: y = \sin(2x - \pi)$ $\quad [\pi]$

j) $f: y = -3\sin(3x)$ $\quad \left[\dfrac{2\pi}{3}\right]$

k) $f: y = -3 + 2\cos\left(\dfrac{x}{2} + \dfrac{\pi}{2}\right)$ $\quad [4\pi]$

l) $f: y = \operatorname{tg}\left(2x - \dfrac{\pi}{4}\right)$ $\quad \left[\dfrac{\pi}{2}\right]$

m) $f: y = \operatorname{cotg}(x + \pi)$ $\quad [\pi]$

n) $f: y = \operatorname{tg}(2x) + \operatorname{cotg}(2x)$ $\quad \left[\dfrac{\pi}{2}\right]$

o) $f: y = e^{\sin x}$ $\quad [2\pi]$

p) $f: y = 5 + \sin(5x)$ $\quad \left[\dfrac{2\pi}{5}\right]$

Příklad 3.7. K daným funkcím sestrojte inverzní funkce a určete příslušné definiční obory D_f a $D_{f^{-1}}$:

a) $f: y = e^{x-1} + 2$ $\left[D_f = I\!R,\ f^{-1}: y = \ln(x-2)+1,\ D_{f^{-1}} = (2, +\infty)\right]$

b) $f: y = 4x + 2$ $\left[D_f = I\!R,\ f^{-1}: y = \dfrac{x-2}{4},\ D_{f^{-1}} = I\!R\right]$

c) $f: y = \dfrac{1-x}{1+x}$ $\left[D_f = I\!R - \{-1\},\ f^{-1}: y = \dfrac{1-x}{1+x},\ D_{f^{-1}} = I\!R - \{-1\}\right]$

d) $f: y = \sqrt{1 + e^{2x}}$ $\left[D_f = I\!R,\ f^{-1}: y = \dfrac{1}{2}\ln(x^2 - 1),\ D_{f^{-1}} = (1, +\infty)\right]$

e) $f: y = \ln(2 - x)$ $\left[D_f = (-\infty, 2),\ f^{-1}: y = 2 - e^x,\ D_{f^{-1}} = I\!R\right]$

f) $f: y = 1 + \sqrt{3 + e^{2x}}$ $\left[D_f = I\!R,\ f^{-1}: y = \dfrac{1}{2}\ln(x^2 - 2x - 2),\ D_{f^{-1}} = (1 + \sqrt{3}, +\infty)\right]$

g) $f: y = \dfrac{4 + e^x}{4 - e^x}$ $\left[D_f = I\!R - \{\ln 4\},\ f^{-1}: y = \ln\left(\dfrac{4x-4}{1+x}\right),\ D_{f^{-1}} = (-\infty, -1) \cup (1, +\infty)\right]$

h) $f: y = \ln(5 - 2x)$ $\left[D_f = \left(-\infty, \dfrac{5}{2}\right),\ f^{-1}: y = \dfrac{5 - e^x}{2},\ D_{f^{-1}} = I\!R\right]$

i) $f: y = \sqrt{3 + e^x}$ $\left[D_f = I\!R,\ f^{-1}: y = \ln\left(x^2 - 3\right),\ D_{f^{-1}} = (\sqrt{3}, +\infty)\right]$

j) $f: y = \dfrac{2 + e^x}{e^x}$ $\left[D_f = I\!R,\ f^{-1}: y = \ln\left(\dfrac{2}{x-1}\right),\ D_{f^{-1}} = (1, +\infty)\right]$

k) $f: y = 1 + \dfrac{1}{x}$ $\left[D_f = I\!R - \{0\},\ f^{-1}: y = \dfrac{1}{x-1},\ D_{f^{-1}} = I\!R - \{1\}\right]$

l) $f: y = 3 - 5x$ $\left[D_f = I\!R,\ f^{-1}: y = \dfrac{3-x}{5},\ D_{f^{-1}} = I\!R\right]$

m) $f: y = x^3 + 2$ $\left[D_f = I\!R,\ f^{-1}: y = \sqrt[3]{x-2},\ D_{f^{-1}} = I\!R\right]$

n*) $f: y = \dfrac{1}{\ln x}$ $\left[D_f = (0, 1) \cup (1, +\infty),\ f^{-1}: y = e^{\frac{1}{x}},\ D_{f^{-1}} = I\!R - \{0\}\right]$

o) $f: y = \dfrac{1 - e^x}{2}$ $\left[D_f = I\!R,\ f^{-1}: y = \ln(1 - 2x),\ D_{f^{-1}} = \left(-\infty, \dfrac{1}{2}\right)\right]$

p) $f: y = e^{-x} + 1$ $\left[D_f = I\!R,\ f^{-1}: y = -\ln(x-1),\ D_{f^{-1}} = (1, +\infty)\right]$

q) $f: y = 2 - \ln x$ $\left[D_f = (0, +\infty),\ f^{-1}: y = e^{2-x},\ D_{f^{-1}} = I\!R\right]$

r) $f: y = 1 - \sqrt{-x}$ $\left[D_f = (-\infty, 0\rangle,\ f^{-1}: y = -(1-x)^2,\ D_{f^{-1}} = (-\infty, 1\rangle\right]$

u) $f: y = -\sqrt{1 - x}$ $\left[D_f = (-\infty, -1\rangle,\ f^{-1}: y = 1 - x^2,\ D_{f^{-1}} = (-\infty, 0\rangle\right]$

v) $f: y = 2\arcsin(x+1)$ $\left[D_f = \langle -2; 0\rangle,\ f^{-1}: y = \sin\dfrac{x}{2} - 1,\ D_{f^{-1}} = \langle -\pi; \pi\rangle\right]$

w) $f: y = \arccos(2x+1)$ $\left[D_f = \langle -1; 1\rangle,\ f^{-1}: y = \dfrac{\cos x - 1}{2},\ D_{f^{-1}} = \langle 0; \pi\rangle\right]$

x) $f: y = 3\arcsin\left(1 - \dfrac{x}{2}\right)$ $\left[D_f = \langle 0; 4\rangle,\ f^{-1}: y = 2 - 2\sin\left(\dfrac{x}{3}\right),\ D_{f^{-1}} = \left\langle -\dfrac{3\pi}{2}; \dfrac{3\pi}{2}\right\rangle\right]$

y) $f: y = \operatorname{arctg}(1 - x)$ $\left[D_f = I\!R,\ f^{-1}: y = 1 - \operatorname{tg} x,\ D_{f^{-1}} = \left(-\dfrac{\pi}{2}; \dfrac{\pi}{2}\right)\right]$

z) $f: y = e^{1-x} + \sqrt{2}$ $\left[D_f = I\!R,\ f^{-1}: y = 1 - \ln\left(x - \sqrt{2}\right),\ D_{f^{-1}} = \left(-\sqrt{2}; +\infty\right)\right]$

Příklad 3.8. K daným funkcím sestrojte inverzní funkce v daném definičním oboru a určete $D_{f^{-1}}$:

a) $f: y = \dfrac{x-1}{x+1},\ D_f = (-\infty; -1)$ $\left[f^{-1}: y = \dfrac{x+1}{1-x},\ D_{f^{-1}} = (1; +\infty)\right]$

b) $f: y = 1 + x^2,\ D_f = (-\infty; 0\rangle$ $\left[f^{-1}: y = -\sqrt{x-1},\ D_{f^{-1}} = \langle 1; +\infty)\right]$

c) $f: y = 1 + \sin x,\ D_f = \left(-\dfrac{\pi}{2}; \dfrac{\pi}{2}\right)$ $\left[f^{-1}: y = \arcsin(x-1),\ D_{f^{-1}} = (0; 2)\right]$

d) $f: y = \sqrt{1-x^2},\ D_f = \langle -1; 0\rangle$ $\left[f^{-1}: y = -\sqrt{1-x^2},\ D_{f^{-1}} = \langle 0; 1\rangle\right]$

e) $f: y = \sqrt{9-x^2},\ D_f = \langle -3; 0\rangle$ $\left[f^{-1}: y = -\sqrt{9-x^2},\ D_{f^{-1}} = \langle 0; 3\rangle\right]$

f) $f: y = \sqrt{9-x^2},\ D_f = \langle 0; 3\rangle$ $\left[f^{-1}: y = \sqrt{9-x^2},\ D_{f^{-1}} = \langle 0; 3\rangle\right]$

g) $f: y = \sin^2 x + \sin x - 2,\ D_f = (0; \pi)$ $[]$

h) $f: y = \cos x,\ D_f = \langle 11\pi; 12\pi\rangle$ $\left[f^{-1}: y = \arcsin x + \dfrac{23\pi}{2},\ D_{f^{-1}} = \langle -1; 1\rangle\right]$

i) $f: y = 4\arcsin\left(\sqrt{1-x^2}\right),\ D_f = \langle 0; 1\rangle$ $\left[f^{-1}: y = \sqrt{1-\sin^2\left(\dfrac{x}{4}\right)},\ D_{f^{-1}} = \langle 0; 2\pi\rangle\right]$

j) $f: y = \ln(\cos x),\ D_f = \left\langle 0; \dfrac{\pi}{4}\right\rangle$ $\left[f^{-1}: y = \arccos\left(e^x\right),\ D_{f^{-1}} = \left\langle \ln\dfrac{\sqrt{2}}{2}; 0\right\rangle\right]$

k) $f: y = 2\sin(3x),\ D_f = \left\langle 0; \dfrac{\pi}{6}\right\rangle$ $\left[f^{-1}: y = \dfrac{1}{3}\arcsin\left(\dfrac{x}{2}\right)\ D_{f^{-1}} = \langle 0; 2\rangle\right]$

l) $f: y = \sqrt{1+\sin x},\ D_f = \left\langle -\dfrac{\pi}{2}; \dfrac{\pi}{2}\right\rangle$ $\left[f^{-1}: y = \arcsin(x^2-1),\ D_{f^{-1}} = \langle 0; \sqrt{2}\rangle\right]$

m) $f: y = \dfrac{1}{5}e^{\operatorname{cotg} x},\ D_f = (0; \pi)$ $\left[f^{-1}: y = \operatorname{arccotg}\left(\ln 5x\right) D_{f^{-1}} = (0; +\infty)\right]$

n) $f: y = \sin x,\ D_f = \left\langle \dfrac{\pi}{2}; \pi\right\rangle$ $\left[f^{-1}: y = \arccos x + \dfrac{\pi}{2},\ D_{f^{-1}} = \langle 0; 1\rangle\right]$

o) $f: y = 1 + 2\operatorname{tg} x,\ D_f = \left(-\dfrac{\pi}{2}; \dfrac{\pi}{2}\right)$ $\left[f^{-1}: y = \operatorname{arctg}\left(\dfrac{x-1}{2}\right)\ D_{f^{-1}} = \mathbb{R}\right]$

p) $f: y = \arcsin(\sin x),\ D_f = \left\langle -\dfrac{\pi}{2}; \dfrac{\pi}{2}\right\rangle$ $\left[f^{-1}: y = \arcsin(\sin x),\ D_{f^{-1}} = \left\langle -\dfrac{\pi}{2}; \dfrac{\pi}{2}\right\rangle\right]$

q) $f: y = 2\sin(4x),\ D_f = \left\langle -\dfrac{\pi}{8}; \dfrac{\pi}{8}\right\rangle$ $\left[f^{-1}: y = \dfrac{1}{4}\arcsin\left(\dfrac{x}{2}\right),\ D_{f^{-1}} = \langle -2; 2\rangle\right]$

r) $f: y = 4\operatorname{tg}(4x),\ D_f = \left(-\dfrac{\pi}{8}; \dfrac{\pi}{8}\right)$ $\left[f^{-1}: y = \dfrac{1}{4}\operatorname{arctg}\left(\dfrac{x}{4}\right),\ D_{f^{-1}} = \mathbb{R}\right]$

Kapitola 4

Limita a spojitost funkce

Příklad 4.1. Vypočítejte limity:

a)	$\lim\limits_{x\to 2}\dfrac{x^2-4x+1}{2x+1}$	$\left[-\dfrac{3}{5}\right]$	b)	$\lim\limits_{x\to \frac{\pi}{4}}\dfrac{1+\sin(2x)}{1-\cos(4x)}$	$[1]$
c)	$\lim\limits_{x\to 2}(x-1)\sin\left(\dfrac{\pi x}{4}\right)$	$[1]$	d)	$\lim\limits_{x\to 4}\log\left(x^2-2x+2\right)$	$[1]$
e)	$\lim\limits_{x\to \frac{\pi}{4}}\dfrac{\sin^4 x}{x}$	$\left[\dfrac{1}{\pi}\right]$	f*)	$\lim\limits_{x\to 1} x\operatorname{arctg} x$	$\left[\dfrac{\pi}{4}\right]$
g*)	$\lim\limits_{x\to 2}\dfrac{x-2}{x^2-3x+2}$	$[1]$	h)	$\lim\limits_{x\to -3}\dfrac{3x^2+11x+6}{x^3+27}$	$\left[-\dfrac{7}{27}\right]$
i)	$\lim\limits_{x\to \pi}\dfrac{\operatorname{tg} x}{\sin(2x)}$	$\left[\dfrac{1}{2}\right]$	j)	$\lim\limits_{x\to \frac{\pi}{4}}\dfrac{\sin x-\cos x}{\cos(2x)}$	$\left[-\dfrac{1}{\sqrt{2}}\right]$
k)	$\lim\limits_{x\to -2}\dfrac{x^2+x-2}{x^2+2x}$	$\left[\dfrac{3}{2}\right]$	l)	$\lim\limits_{x\to -2}\dfrac{3x^2+3x-6}{2x^2-2x-12}$	$\left[\dfrac{9}{10}\right]$
m)	$\lim\limits_{x\to -2}\dfrac{x^3+3x^2+2x}{x^2-x-6}$	$\left[-\dfrac{2}{5}\right]$	m)	$\lim\limits_{x\to 1}\dfrac{x^2-3x+2}{x^2-1}$	$\left[-\dfrac{1}{2}\right]$
o*)	$\lim\limits_{x\to 2}\left(\dfrac{1}{2-x}-\dfrac{x+10}{8-x^3}\right)$	$\left[-\dfrac{5}{12}\right]$	p)	$\lim\limits_{x\to 6}\dfrac{x-6}{\sqrt{x+3}-3}$	$[6]$
q*)	$\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$	$[4]$	r*)	$\lim\limits_{x\to -1}\dfrac{x^3+1}{\sqrt{x^2-3x}+2x}$	$[4]$
s)	$\lim\limits_{x\to 1}\dfrac{2-\sqrt{x+3}}{x^3-1}$	$\left[-\dfrac{1}{12}\right]$	t)	$\lim\limits_{x\to 5}\dfrac{\sqrt{x-1}-2}{x^2-4x-5}$	$\left[\dfrac{1}{24}\right]$
u)	$\lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2}-1}{x}$	$[0]$	v)	$\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+x+1}-1}{x}$	$\left[\dfrac{1}{2}\right]$
w)	$\lim\limits_{x\to 0}\dfrac{\operatorname{tg} x-\sin x}{\sin^3 x}$	$\left[\dfrac{1}{2}\right]$	x)	$\lim\limits_{x\to 1}\dfrac{x^2-x}{\sqrt{x}-1}$	$[2]$
y)	$\lim\limits_{x\to 2}\dfrac{x^3-8}{x^4-16}$	$\left[\dfrac{3}{8}\right]$	z)	$\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}$	$\left[\dfrac{1}{2}\right]$

Příklad 4.2. Vypočítejte limity:

a) $\lim\limits_{x\to+\infty} \dfrac{x^3+x}{x^4+3x^2+1}$ $[0]$

b) $\lim\limits_{x\to+\infty} \dfrac{3x^2+1}{5x^2-x+1}$ $\left[\dfrac{3}{5}\right]$

c) $\lim\limits_{x\to+\infty} \dfrac{x^4-x^2+2}{x^3-x^2+1}$ $[+\infty]$

d) $\lim\limits_{x\to+\infty} \dfrac{x^4+3x-1}{2x^2+5}$ $[+\infty]$

e) $\lim\limits_{x\to+\infty} \dfrac{x^3-2x+1}{x^5-2x+2}$ $[0]$

f) $\lim\limits_{x\to+\infty} \dfrac{2x^3-2x^2+1}{5x^3+2x-1}$ $\left[\dfrac{2}{5}\right]$

g) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+1}}{x}$ $[-1]$

h*) $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+4}+x\right)$ $[0]$

i*) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+3x}}{\sqrt[3]{x^3-2x^2}}$ $[-1]$

j) $\lim\limits_{x\to+\infty} \left(\sqrt{x-\sqrt{x}}-\sqrt{x}\right)$ $\left[-\dfrac{1}{2}\right]$

k) $\lim\limits_{x\to+\infty} \dfrac{x^2-x+1}{2x^2+x-3}$ $\left[\dfrac{1}{2}\right]$

l) $\lim\limits_{x\to+\infty} \dfrac{2x^2+3}{\sqrt{3x^4-1}}$ $\left[\dfrac{2\sqrt{3}}{3}\right]$

m) $\lim\limits_{x\to-\infty} x\left(\sqrt{x^2+9}-\sqrt{x^2-9}\right)$ $[-9]$

n) $\lim\limits_{x\to+\infty} \dfrac{3x^3-4x^2+2}{7x^3+5x^2-3}$ $\left[\dfrac{3}{7}\right]$

o) $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x^2+3x}}{\sqrt{2x^3-2x}}$ $[0]$

p) $\lim\limits_{x\to+\infty} x\left(\sqrt{1+x^2}-x\right)$ $\left[\dfrac{1}{2}\right]$

q*) $\lim\limits_{x\to+\infty} \left(\dfrac{2x+3}{2x+1}\right)^{x+1}$ $[e]$

r) $\lim\limits_{x\to+\infty} \operatorname{arctg} x$ $\left[\dfrac{\pi}{2}\right]$

s) $\lim\limits_{x\to+\infty} \dfrac{2}{1+e^{1/x}}$ $[1]$

t) $\lim\limits_{x\to+\infty} \operatorname{arccotg} x$ $[0]$

Příklad 4.3. Vypočítejte limity:

a) $\lim\limits_{x\to 0} x\sin\dfrac{1}{x}$ $[0]$

b) $\lim\limits_{x\to+\infty} \dfrac{\cos e^{x^2+x+1}}{x}$ $[0]$

c) $\lim\limits_{x\to+\infty} \dfrac{\sin(5x)}{2x}$ $[0]$

d) $\lim\limits_{x\to 0} x\cos\dfrac{1}{x^2}$ $[0]$

e) $\lim\limits_{x\to-\infty} \dfrac{\cos(4x-\pi)}{2x}$ $[0]$

f) $\lim\limits_{x\to 0} \operatorname{arctg}(-x)\cdot\sin\left(\dfrac{2}{27x^3}\right)$ $[0]$

Příklad 4.4. Vypočítejte limity:

a) $\lim\limits_{x\to 0} \dfrac{\operatorname{tg} x}{x}$ $[1]$

b*) $\lim\limits_{x\to 0} \dfrac{\sin x - x}{\sin x + x}$ $[0]$

c) $\lim\limits_{x\to 0} \dfrac{1-\cos(2x)+\operatorname{tg}^2 x}{x\sin x}$ $[3]$

d) $\lim\limits_{x\to 0} \cos\left(x^2\sin\dfrac{1}{x}\right)$ $[1]$

e) $\lim\limits_{x\to 0} \dfrac{\sin(5x)}{x}$ $[5]$

f) $\lim\limits_{x\to 0} \dfrac{\arcsin(3x^2)}{x\cdot\operatorname{tg}(3x)}$ $[1]$

g*) $\lim\limits_{x\to 0} \dfrac{\sin(7x)}{\sin(4x)}$ $\left[\dfrac{7}{4}\right]$

h) $\lim\limits_{x\to 0} \dfrac{\operatorname{tg}(x^2)}{x\sin(3x)}$ $\left[\dfrac{1}{3}\right]$

i) $\lim\limits_{x\to 0} \dfrac{\arcsin x}{x}$ $[1]$

j) $\lim\limits_{x\to 0} \dfrac{\arcsin^4(2x)}{x\operatorname{tg}(2x^3)}$ $[8]$

Příklad 4.5. Vypočítejte limity:

a) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{8+x}-2}{x}$ $\left[\dfrac{1}{12}\right]$

b) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{27+x}-3}{x}$ $\left[\dfrac{1}{27}\right]$

c) $\lim\limits_{x\to 0} \dfrac{x}{1-\sqrt[3]{x+1}}$ $[-3]$

d) $\lim\limits_{x\to 0} \dfrac{5x}{1-\sqrt[3]{x+1}}$ $[-15]$

e) $\lim\limits_{x\to 0} \dfrac{x}{\sqrt[3]{1+x}-1}$ $[3]$

f) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+x}-1}{x}$ $\left[\dfrac{1}{3}\right]$

Příklad 4.6. Existují-li následující limity, určete jejich hodnotu:

a*)	$\lim\limits_{x\to 0}\dfrac{x^2-1}{x^2}$	$[-\infty]$	b*)	$\lim\limits_{x\to -1}\dfrac{2x}{x+1}$	[neex.]
c)	$\lim\limits_{x\to 0}\left(\dfrac{4x+4}{x^2}-2\right)$	$[+\infty]$	d)	$\lim\limits_{x\to 2}\dfrac{x^2-1}{x^2-2x}$	[neex.]
e)	$\lim\limits_{x\to 0}\dfrac{\ln(x+2)}{\operatorname{arctg} x}$	[neex.]	f)	$\lim\limits_{x\to 2}\dfrac{x}{x^2-4}$	[neex.]
g)	$\lim\limits_{x\to 0}\dfrac{\sin x+1}{\sin x}$	[neex.]	h)	$\lim\limits_{x\to 0}\dfrac{\cos x+1}{\cos x-1}$	[neex.]
i)	$\lim\limits_{x\to -2}\dfrac{x^3}{(x+2)^2}$	$[+\infty]$	j)	$\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}+\sqrt{x}}{x}$	[neex.]

Příklad 4.7. Vypočítejte jednostranné limity:

a)	$\lim\limits_{x\to 1+}\dfrac{1}{1-x}$	$[-\infty]$	b)	$\lim\limits_{x\to 1-}\dfrac{1}{1-x}$	$[+\infty]$
c)	$\lim\limits_{x\to 1+}\dfrac{x^2-1}{x-1}$	[2]	d)	$\lim\limits_{x\to 1-}\dfrac{x^2-1}{x-1}$	[2]
e)	$\lim\limits_{x\to -1+}\dfrac{1}{x+1}$	$[+\infty]$	f)	$\lim\limits_{x\to -1-}\dfrac{1}{x+1}$	$[-\infty]$
g)	$\lim\limits_{x\to 2+}\dfrac{\lvert x-2\rvert}{x-2}$	[1]	h)	$\lim\limits_{x\to 2-}\dfrac{\lvert x-2\rvert}{x-2}$	$[-1]$
i)	$\lim\limits_{x\to 2+}\dfrac{x}{x-2}$	$[+\infty]$	j)	$\lim\limits_{x\to \pi+}\dfrac{1}{\sin x}$	$[-\infty]$
k)	$\lim\limits_{x\to 1-}\operatorname{arctg}\left(\dfrac{1}{1-x}\right)$	$\left[\dfrac{\pi}{2}\right]$	l)	$\lim\limits_{x\to \frac{\pi}{2}-}\operatorname{tg} x$	$[+\infty]$

Příklad 4.8. Najděte všechny asymptoty grafu funkcí:

a) $f(x) = \dfrac{1}{x^2}$ $\quad [x = 0, y = 0]$

b) $f(x) = 5x + \dfrac{\sin x}{x}$ $\quad [y = 5x]$

c) $f(x) = \dfrac{3x^2}{x-1}$ $\quad [x = 1, y = 3x + 3]$

d) $f(x) = \dfrac{4 + x^3}{4 - x^2}$ $\quad [x = \pm 2, y = -x]$

e) $f(x) = \dfrac{e^x}{x+1}$ $\quad [x = -1, y = 0]$

f) $f(x) = \dfrac{x^3}{x^2 - 1}$ $\quad [x = \pm 1, y = x]$

g) $f(x) = x - \operatorname{arctg} x$ $\quad \left[y = x \pm \dfrac{\pi}{2}\right]$

h) $f(x) = \dfrac{1}{x^2 - 9}$ $\quad [x = \pm 3, y = 0]$

i) $f(x) = \dfrac{\cos x}{x}$ $\quad [x = 0, y = 0]$

j) $f(x) = x + \dfrac{\ln x}{x}$ $\quad [x = 0, y = x]$

k) $f(x) = \dfrac{x}{x-1}$ $\quad [x = 1, y = 1]$

l) $f(x) = 3x + \dfrac{3}{x-2}$ $\quad [x = 2, y = 3x]$

m) $f(x) = x + \dfrac{2x}{x^2 - 1}$ $\quad [x = \pm 1, y = x]$

n) $f(x) = xe^{\frac{1}{x^2}}$ $\quad [x = 0, y = x]$

o) $f(x) = \sqrt[3]{x^3 + 4x^2}$ $\quad \left[y = x + \dfrac{4}{3}\right]$

p) $f(x) = x + \ln\left(e + \dfrac{1}{x}\right)$ $\quad \left[x = -\dfrac{1}{e}, y = x + \dfrac{1}{e}\right]$

q) $f(x) = \dfrac{2x^3 + x^2 - 2x}{x^2 - 1}$ $\quad [x = \pm 1, y = 2x + 1]$

r) $f(x) = \dfrac{4x^2}{\sqrt{x^2 - 1}}$ $\quad [x = \pm 1, y = \pm 4x]$

s) $f(x) = 2x - \arccos \dfrac{1}{x}$ $\quad \left[y = 2x - \dfrac{\pi}{2}\right]$

t) $f(x) = xe^{\frac{1}{x}}$ $\quad [x = 0, y = x + 1]$

u) $f(x) = \operatorname{arctg} x + 3x$ $\quad \left[y = 3x \pm \dfrac{\pi}{2}\right]$

v) $f(x) = \sqrt{x^2 + x^{3/2}}$ $\quad$ [neex.]

w) $f(x) = x + e^{-x}$ $\quad [y = x]$

x) $f(x) = \arccos\left(\dfrac{1 - x^2}{1 + x^2}\right)$ $\quad [y = \pi]$

y) $f(x) = \operatorname{arctg}\left(\dfrac{1}{x}\right)$ $\quad [y = 0]$

z) $f(x) = \operatorname{arctg}\left(\dfrac{x}{x-1}\right)$ $\quad \left[y = \dfrac{\pi}{4}\right]$

Kapitola 5

Derivace funkce a geometrický význam derivace

Příklad 5.1. Vypočtěte f' (výsledek upravte) a určete D_f a $D_{f'}$, jsou-li funkce f zadány předpisem:

a) $f(x) = \dfrac{5}{3}x - 2 + \dfrac{2}{3x^2}$ $\left[\dfrac{5x^3-4}{3x^3}\right]$

b) $f(x) = 6\sqrt[3]{x} - 4\sqrt[4]{x} + \dfrac{5}{3\sqrt[3]{x}}$ $\left[\dfrac{2}{\sqrt[3]{x^2}} - \dfrac{1}{\sqrt[4]{x^3}} - \dfrac{5}{9\sqrt[3]{x^4}}\right]$

c) $f(x) = \dfrac{5}{3\sqrt[4]{x^3}}$ $\left[\dfrac{-5}{4\sqrt[4]{x^7}}\right]$

d) $f(x) = 3x^3 - 2\sqrt{x} + \dfrac{1}{x^3}$ $\left[9x^2 - \dfrac{1}{\sqrt{x}} - \dfrac{3}{x^4}\right]$

e) $f(x) = 3\sqrt[3]{x^2} - \dfrac{1}{3}\operatorname{cotg} x$ $\left[\dfrac{2}{\sqrt[3]{x}} + \dfrac{1}{3\sin^2 x}\right]$

f) $f(x) = 2x^3 + 5\sin x$ $\left[6x^2 + 5\cos x\right]$

g) $f(x) = (x^2-1)(x^3-5)$ $\left[5x^4 - 3x^2 - 10x\right]$

h) $f(x) = x^2 \operatorname{tg} x$ $\left[\dfrac{x(x+\sin(2x))}{\cos^2 x}\right]$

i) $f(x) = (x^2+1)\ln x$ $\left[2x\ln x + x + \dfrac{1}{x}\right]$

j) $f(x) = x^2 \operatorname{cotg} x$ $\left[2x\operatorname{cotg} x - \dfrac{x^2}{\sin^2 x}\right]$

k) $f(x) = \sqrt{x}\cos x$ $\left[\dfrac{\cos x}{2\sqrt{x}} - \sqrt{x}\sin x\right]$

l) $f(x) = \sin x \cos x$ $[\cos(2x)]$

m) $f(x) = \sqrt[3]{x^2}\operatorname{arctg} x$ $\left[\dfrac{2\operatorname{arctg} x}{3\sqrt[3]{x}} + \dfrac{\sqrt[3]{x^2}}{1+x^2}\right]$

n) $f(x) = 2^x \log_2 x$ $\left[2^x\left(\ln 2 \cdot \log_2 x + \dfrac{1}{x\ln 2}\right), x > 0\right]$

o) $f(x) = \pi\sqrt{x}\arcsin x$ $\left[\dfrac{\pi\arcsin x}{2\sqrt{x}} + \dfrac{\pi\sqrt{x}}{\sqrt{1-x^2}}, x \in (0,1)\right]$

p) $f(x) = \dfrac{x^3}{3}(e^x + e)$ $\left[x^2\left(e^x\left(1+\dfrac{x}{3}\right)+e\right), x \in I\!R\right]$

q) $f(x) = x^2 \arccos x$ $\left[x\left(2\arccos x - \dfrac{x}{\sqrt{1-x^2}}\right)\right]$

r) $f(x) = xe^x \cos x$ $\left[e^x(\cos x + x(\cos x - \sin x))\right]$

s) $f(x) = x^3\sqrt{x}e^x$ $\left[e^x\left(\dfrac{7}{2}\sqrt{x^5} + \sqrt{x^7}\right),\ x \geq 0\right]$

Příklad 5.2. Vypočtěte f' (výsledek upravte) a určete D_f a $D_{f'}$, jsou-li funkce f zadány předpisem:

a) $f(x) = \dfrac{x}{x+1}$ $\left[\dfrac{1}{(1+x)^2}\right]$

b) $f(x) = \dfrac{\cos x}{1-\sin x}$ $\left[\dfrac{1}{1-\sin x}\right]$

c) $f(x) = \dfrac{e^x}{\sin x}$ $\left[\dfrac{e^x(\sin x - \cos x)}{\sin^2 x}\right]$

d) $f(x) = \dfrac{1+x-x^2}{1-x+x^2}$ $\left[\dfrac{2(1-2x)}{(1-x+x^2)^2}\right]$

e) $f(x) = \dfrac{x+\sqrt[3]{x}}{x-\sqrt[3]{x}}$ $\left[\dfrac{-4\sqrt[3]{x}}{3(x-\sqrt[3]{x})^2}\right]$

f) $f(x) = \dfrac{\operatorname{arctg} x}{\log x}$ $\left[\dfrac{x\ln(10)\log x - (1+x^2)\operatorname{arctg} x}{(x+x^3)\ln(10)\log^2 x}\right]$

g) $f(x) = \dfrac{xe^x}{1+x^2}$ $\left[\dfrac{e^x(1+x-x^2+x^3)}{(1+x^2)^2}\right]$

h) $f(x) = \dfrac{(x^2+1)\operatorname{arctg} x}{\ln x}$ $\left[\dfrac{x(2x\operatorname{arctg} x+1)\ln x - (x^2+1)\operatorname{arctg} x}{x\ln^2 x}\right]$

i) $f(x) = \dfrac{x^2\ln x}{x+1}$ $\left[\dfrac{(2\ln x+1)(x^2+x)-x^2\ln x}{(x+1)^2}\right]$

j) $f(x) = 2e^{3x}$ $\left[6e^{3x}\right]$

k) $f(x) = 3\ln(5x)$ $\left[\dfrac{3}{x}\right]$

l) $f(x) = \ln(x^2-1)$ $\left[\dfrac{2x}{x^2-1}\right]$

m) $f(x) = \arcsin\left(\dfrac{x-2}{2}\right)$ $\left[\dfrac{1}{\sqrt{4x-x^2}}\right]$

n) $f(x) = \operatorname{arctg}\left(\dfrac{1+x}{1-x}\right)$ $\left[\dfrac{1}{1+x^2}\right]$

o) $f(x) = \dfrac{1}{(x^3-1)^2}$ $\left[\dfrac{-6x^2}{(x^3-1)^3}\right]$

p) $f(x) = 4\ln\left(\dfrac{2}{x}\right)$ $\left[-\dfrac{4}{x}, x>0\right]$

q) $f(x) = \dfrac{-1}{\sqrt[3]{1+x^2}}$ $\left[\dfrac{2x}{3\cdot\sqrt[3]{(1+x^2)^4}}, x\in\mathbb{R}\right]$

r) $f(x) = \ln\left(\dfrac{1}{(2-x)^2}\right)$ $\left[\dfrac{2}{2-x}, x\neq 2\right]$

s) $f(x) = \ln\left(\dfrac{x+1}{x}\right)$ $\left[\dfrac{-1}{x^2+x},\ x\in(-\infty;-1)\cup(0;+\infty)\right]$

Příklad 5.3. Vypočtěte f' (výsledek upravte) a určete D_f a $D_{f'}$, jsou-li funkce f zadány předpisem:

a) $f(x) = \dfrac{\mathrm{tg}^2 x}{2} + \ln(\cos x)$ $\quad [\mathrm{tg}^3 x]$

b) $f(x) = \arccos(1 - x^2)$ $\quad \left[\dfrac{2\mathrm{sgn}\, x}{\sqrt{2 - x^2}}\right]$

c) $f(x) = \ln(4 - x^2) + \arcsin\left(\dfrac{x-2}{2}\right)$ $\quad \left[\dfrac{2x}{x^2 - 4} + \dfrac{1}{\sqrt{4x - x^2}}\right]$

d) $f(x) = \ln(1 + \cos x)$ $\quad \left[\dfrac{-\sin x}{1 + \cos x}\right]$

e) $f(x) = \mathrm{arctg}\sqrt{6x - 1}$ $\quad \left[\dfrac{1}{2x\sqrt{6x - 1}}\right]$

f) $f(x) = 2\arcsin\sqrt{\dfrac{x}{2}} - \sqrt{2x - x^2}$ $\quad \left[\sqrt{\dfrac{x}{2 - x}}\right]$

g) $f(x) = (x^2 + 4)\mathrm{arctg}\,\dfrac{x}{2} - 2x$ $\quad \left[2x\mathrm{arctg}\,\dfrac{x}{2}\right]$

h) $f(x) = (x - 2)\sqrt{1 + e^x} - \ln\left(\dfrac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1}\right)$ $\quad \left[\dfrac{xe^x}{2\sqrt{1 + e^x}}\right]$

i) $f(x) = \sqrt{\dfrac{1 - e^x}{1 + e^x}}$ $\quad \left[\dfrac{-e^x}{(1 + e^x)\sqrt{1 - e^{2x}}}\right]$

j) $f(x) = 2\sqrt{x + 1} + \ln\left(\dfrac{x + 2 - 2\sqrt{x + 1}}{x}\right)$ $\quad \left[\dfrac{\sqrt{1 + x}}{x}\right]$

k) $f(x) = \ln(e^x + \sqrt{1 + e^{2x}})$ $\quad \left[\dfrac{e^x}{\sqrt{1 + e^{2x}}}\right]$

l) $f(x) = \left(\dfrac{1}{1 - x}\right)^x$ $\quad \left[\left(\dfrac{1}{1 - x}\right)^x \left(\dfrac{x}{1 - x} - \ln(1 - x)\right)\right]$

m) $f(x) = (x^2 + 1)^{\mathrm{arctg}\, x}$ $\quad \left[(x^2 + 1)^{\mathrm{arctg}\, x - 1}\left(2x\mathrm{arctg}\, x + \ln(x^2 + 1)\right)\right]$

n) $f(x) = x^{\sin x}$ $\quad \left[x^{\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x}\right)\right]$

o) $f(x) = (2x)^{3x+1}$ $\quad \left[(2x)^{3x+1}\left(3\ln(2x) + \dfrac{3x + 1}{x}\right),\ x > 0\right]$

Příklad 5.4. Vypočtěte druhou derivaci f'' (výsledek upravte), jsou-li funkce f zadány předpisem:

a) $f(x) = \sin^2 x$ $\quad [2\cos(2x)]$

b) $f(x) = \mathrm{tg}^2 x$ $\quad \left[\dfrac{2\cos^2 x + 6\sin^2 x}{\cos^4 x}\right]$

c) $f(x) = \sqrt{1 + x^2}$ $\quad \left[\dfrac{1}{\sqrt{(1 + x^2)^3}}\right]$

d) $f(x) = \ln(1 + \cos x)$ $\quad \left[\dfrac{-1}{1 + \cos x}\right]$

e) $f(x) = \ln(x^2 + 1)$ $\quad \left[\dfrac{2 - 2x^2}{(x^2 + 1)^2}\right]$

Příklad 5.5. Vypočtěte třetí derivaci f''' (výsledek upravte), jsou-li funkce f zadány předpisem:

a) $f(x) = \cos^2 x$ $\quad [4\sin(2x)]$

b) $f(x) = x\ln x$ $\quad \left[-\dfrac{1}{x^2}\right]$

c) $f(x) = xe^{2x}$ $\quad [(12 + 8x)e^{2x}]$

Příklad 5.6. Určete rovnici tečny a normály v dotykovém bodě T ke grafu funkcí f daných předpisy:

a) $f(x) = \dfrac{8}{4+x^2}$, $T[2, ?]$ $\quad [t : x + 2y - 4 = 0,\ n : 2x - y - 3 = 0]$

b) $f(x) = \operatorname{arctg}\sqrt{x^2-1}$, $T[\sqrt{2}, ?]$ $\quad \left[t : -\sqrt{2}x + 2y + \dfrac{4-\pi}{2} = 0\right]$

c) $f(x) = 4 - x^2$, $T[2, ?]$ $\quad [t : 4x + y - 8 = 0,\ n : x - 4x - 2 = 0]$

d) $f(x) = \operatorname{arccotg}\left(\dfrac{1-\ln x}{1+\ln x}\right)$, $T[1, ?]$ $\quad \left[t : y - x + 1 - \dfrac{\pi}{4} = 0\right]$

e) $f(x) = \dfrac{1}{x}\ln\dfrac{1}{x}$, $T[1, ?]$ $\quad [t : y + x - 1 = 0]$

f) $f(x) = (x^2-1)^{\sin x}$, $T[\pi, ?]$ $\quad \left[t : y + x\ln(\pi^2-1) + \pi\ln(\pi^2-1) - 1 = 0\right]$

g) $f(x) = (\cos x)^{\cosh x} + 3x$, $T[0, ?]$ $\quad [t : y - 3x - 1 = 0]$

h) $f(x) = \ln x$, $T[e, ?]$ $\quad [t : x - ey = 0,\ n : ex + y - 1 - e^2 = 0]$

i) $f(x) = \left(\dfrac{1}{1-x^2}\right)^{\cosh x} + e^{2x}$, $T[0, ?]$ $\quad \left[n : y + \dfrac{x}{2} + 2 = 0\right]$

j) $f(x) = (\sin x)^{2x} + x^2$, $T\left[\dfrac{\pi}{2}, ?\right]$ $\quad \left[n : y + \dfrac{x}{\pi} - \dfrac{\pi^2}{4} - \dfrac{3}{2} = 0\right]$

k) $f(x) = (\cos x)^{\cosh x} + 3x$, $T\,[0, ?]$ $\quad \left[n : y + \dfrac{x}{3} - 1 = 0\right]$

l) $f(x) = \ln\left(\dfrac{x}{\sqrt{1-x^2}}\right)^{2x} + x^2$, $T\left[\dfrac{1}{2}, ?\right]$ $\quad \left[n : y + \dfrac{3x}{8} - \dfrac{3}{16} + \dfrac{\ln 3}{2} = 0\right]$

m) $f(x) = (\sin x)^{x^2} + 3\cos x$, $T\left[\dfrac{\pi}{2}, ?\right]$ $\quad \left[n : y - \dfrac{x}{3} + \dfrac{\pi}{6} - 1 = 0\right]$

n) $f(x) = \left(4 - x^2\right)^{\sin x} + 3\cos x$, $T\,[0, ?]$ $\quad \left[n : y + \dfrac{x}{\ln 4} - 4 = 0\right]$

Příklad 5.7. Určete rovnice tečen ke grafu funkce f, které jsou rovnoběžné s přímkou p:

a) $f(x) = x^3 - 12x$, $p : y = 2$ $\quad [y = \pm 16]$

b) $f(x) = x^3 + x - 2$, $p : y = 4x - 1$ $\quad [y = 4x - 4, y = 4x + 15]$

c) $f(x) = x^2 + 4x - 5$, $p : x + 4y = 0$ $\quad \left[y = -\dfrac{1}{4}x - \dfrac{609}{64}\right]$

Příklad 5.8. Určete rovnice tečen ke grafu funkce f, které jsou kolmé k přímce p:

a) $f(x) = \dfrac{x^3}{6} + 2$, $p : x + 2y + 3 = 0$ $\quad \left[y = 2x - \dfrac{5}{3}, y = 2x + \dfrac{11}{3}\right]$

b) $f(x) = x^3 + 3x^2 + 5$, $p : 2x - 6y + 1 = 0$ $\quad [y = -3x - 6]$

c) $f(x) = \ln x$, $p : y - 2x - 1$ $\quad \left[y = \dfrac{x}{2} - 1 + \ln 2\right]$

d) $f(x) = -\sqrt{x} + 2$, $p : y = -\dfrac{x}{2} + 4$ $\quad$ [nemá řešení]

Příklad 5.9. Určete derivaci $y'(x)$ funkce definované parametricky následujícími rovnicemi:

a) $x = \operatorname{tg} t$, $y = \dfrac{\sin 2t}{2}$, $t \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ $\quad \left[\dfrac{1 - \operatorname{tg}^2 t}{(1 + \operatorname{tg}^2 t)^2}\right]$

b) $x = a\cos^3 t$, $y = a\sin^3 t$, $t \in (0, 2\pi)$ $\quad [-\operatorname{tg} t]$

Příklad 5.10. Určete derivace $y'(x)$ a $y''(x)$ funkce definované parametricky následujícími rovnicemi:

a) $x = a\cos t$, $y = b\sin t$, $t \in (0, 2\pi)$ $\quad \left[y' = -\dfrac{b}{a}\operatorname{cotg} t,\ y'' = -\dfrac{b}{a^2\sin^3 t}\right]$

Kapitola 6

Diferenciál, Taylorův polynom a další užití derivace

Příklad 6.1. Najděte diferenciál $\mathrm{d}f(x_0;h)$ v příslušném bodě x_0 pro následující funkce:

a)	$f(x) = (\sin x)^{x^2} + x,\ x_0 = \dfrac{\pi}{2}$	$[h]$
b)	$f(x) = \left(\dfrac{1}{1-x^2}\right)^{\cosh x} + e^{2x},\ x_0 = 0$	$[2h]$
c)	$f(x) = \left(\dfrac{1}{e^2-x^2}\right)^{\cosh x} + e^{-3x},\ x_0 = 0$	$[-3h]$
d)	$f(x) = (\sin x)^{x} + 2x,\ x_0 = \dfrac{\pi}{2}$	$[2h]$
e)	$f(x) = \sqrt{x^2+1}$	[]
f)	$f(x) = 4x^2 + \sqrt[3]{x}$	[]
g)	$f(x) = \ln(x + \sqrt{1+x^2})$	[]
h)	$f(x) = \sqrt{\dfrac{1+x}{1-x}}$	[]
i)	$f(x) = \sin^4 x$	[]
j)	$f(x) = \arcsin \dfrac{1}{x}$	[]
k)	$f(x) = \dfrac{x^2}{x^2-1}$	[]

Příklad 6.2. Pomocí diferenciálu určete přibližnou hodnotu (na 4 platné cifry):

a)	$\sqrt{80}$	$[\doteq 8.9445]$	b)	$\ln(1.1)$	$[\doteq 0.1]$	c*)	$2^{1.003}$	$[\doteq 2.004]$
d)	$\sqrt[3]{28}$	$[\doteq 3.037]$	e)	$e^{0.1}$	$[\doteq 1.1]$	f)	$3^{0.997}$	$[\doteq 2.990]$
g*)	$\sqrt[4]{82}$	$[\doteq 3.009]$	h)	$\sqrt{382}$	$[\doteq 19.55]$	i)	$\sin(-0.01)$	$[\doteq -0.01]$
i)	$\sqrt[3]{9}$	$[\doteq 2.083]$	j)	$\sqrt[5]{1.5}$	$[\doteq 1.1]$	k)	$\operatorname{arctg}(1.1)$	$[\doteq 0.8353]$
m)	$\sqrt[4]{267}$	$[\doteq 4.043]$	n)	1.04^5	$[\doteq 1.2]$	k)	$\sqrt[3]{30}$	$[\doteq 3.111]$

Příklad 6.3. Napište Taylorův polynom n-tého stupně $T_n(x)$ v okolí bodu x_0 pro následují funkce:

a) $f: y = \ln x,\ x_0 = 1,\ n = 3$ $\left[\frac{1}{3}(x-1)^3 - \frac{1}{2}(x-1)^2 + (x-1)\right]$

b) $f: y = \cos\frac{x}{2},\ x_0 = \frac{\pi}{2},\ n = 3$ $\left[\frac{\sqrt{2}}{192}\left(x-\frac{\pi}{2}\right)^3 - \frac{\sqrt{2}}{16}\left(x-\frac{\pi}{2}\right)^2 - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{2}\right) + \frac{\sqrt{2}}{2}\right]$

c) $f: y = e^{2x},\ x_0 = 1,\ n = 3$ $\left[\frac{4}{3}e^2(x-1)^3 + 2e^2(x-1)^2 + 2e^2(x-1) + e^2\right]$

d) $f: y = \frac{1+x}{1-x},\ x_0 = 0,\ n = 3$ $\left[1 + 2x + 2x^2 + 2x^3\right]$

Příklad 6.4. Napište Maclaurinův polynom n-tého stupně ($n \in \mathbb{N}$), funkce:

a) $f: y = e^x$ $\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}\right]$

b) $f: y = \cos x$ $\left[1 - \frac{x^2}{4} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^m \frac{x^{2m}}{(2m)!},\quad n = 2m\right]$

c) $f: y = \sin x$ $\left[x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{m+1} \frac{x^{2m-1}}{(2m-1)!},\quad n = 2m-1\right]$

d) $f: y = \ln(1+x)$ []

e) $f: y = \frac{1}{1-x}$ []

f) $f: y = e^{x^2}$ []

g) $f: y = \ln\left(\frac{1+x}{1-x}\right)$ []

Příklad 6.5. Rozviňte podle mocnin $x - x_0$ následující polynomy:

a) $f: y = x^3 - 2x + 5,\ x_0 = 1$ $\left[(x-1)^3 + 3(x-1)^2 + (x-1) + 4\right]$

b) $f: y = x^4 - 3x^2 - 10x - 11,\ x_0 = 2$ $\left[(x-2)^4 + 8(x-2)^3 + 21(x-2)^2 + 10(x-2) - 27\right]$

c) $f: y = x^4 - 2x^2 + x + 2,\ x_0 = -1$ $\left[(x+1)^4 - 4(x+1)^3 + 4(x+1)^2 + (x+1)\right]$

Příklad 6.6. Pomocí l'Hospitalova pravidla vypočítejte limity:

a)	$\lim\limits_{x\to 0} \dfrac{e^x - 1}{\sin(2x)}$	$\left[\dfrac{1}{2}\right]$	b)	$\lim\limits_{x\to 1} \dfrac{x-1}{\ln x}$	$[1]$
c*)	$\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$	$\left[\dfrac{1}{2}\right]$	d)	$\lim\limits_{x\to 3} \dfrac{x-3}{x^2-8x+15}$	$\left[-\dfrac{1}{2}\right]$
e*)	$\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$	$\left[\dfrac{1}{6}\right]$	f)	$\lim\limits_{x\to 0} \dfrac{e^{2x}-2x-1}{\sin^2(3x)}$	$\left[\dfrac{2}{9}\right]$
g*)	$\lim\limits_{x\to +\infty} \dfrac{e^x}{x^3}$	$[+\infty]$	h)	$\lim\limits_{x\to 0+} \dfrac{\ln x}{\operatorname{cotg} x}$	$[0]$
i)	$\lim\limits_{x\to +\infty} \dfrac{x^3-1}{\ln x}$	$[+\infty]$	j)	$\lim\limits_{x\to +\infty} xe^{-x}$	$[0]$
k)	$\lim\limits_{x\to 0} \left(\dfrac{1}{x\sin x} - \dfrac{1}{x^2}\right)$	$\left[\dfrac{1}{6}\right]$	l)	$\lim\limits_{x\to 1-} \ln x \cdot \ln(1-x)$	$[0]$
m)	$\lim\limits_{x\to +\infty} (\pi - 2\operatorname{arctg} x)\ln x$	$[0]$	n)	$\lim\limits_{x\to \pm\infty} x(e^{\frac{1}{x}} - 1)$	$[1]$
o)	$\lim\limits_{x\to 0+} \left(\dfrac{1}{x} - \operatorname{cotg} x\right)$	$[0]$	p)	$\lim\limits_{x\to 0+} (\sin x)^{\operatorname{tg} x}$	$[1]$
q*)	$\lim\limits_{x\to 0+} \left(\dfrac{1}{x}\right)^{\operatorname{tg} x}$	$[1]$	r)	$\lim\limits_{x\to 0+} (1+3\operatorname{tg}^2 x)^{\operatorname{cotg}(2x)}$	$[e^3]$
s)	$\lim\limits_{x\to \pm\infty} \left(\dfrac{x^2-1}{x^2}\right)^{x^4}$	$[0]$	t)	$\lim\limits_{x\to 0} (\cos(3x))^{\frac{1}{x^2}}$	$\left[e^{-\frac{9}{2}}\right]$
u)	$\lim\limits_{x\to 0} \left(\operatorname{tg}\left(\dfrac{\pi}{4} + x\right)\right)^{\operatorname{cotg}(2x)}$	$[e]$	v)	$\lim\limits_{x\to 1} \dfrac{\ln x}{1-x}$	$[-1]$
w)	$\lim\limits_{x\to 1+} x^{\frac{1}{1-x}}$	$\left[\dfrac{1}{e}\right]$	x)	$\lim\limits_{x\to 0} \dfrac{2\arcsin x}{3x}$	$\left[\dfrac{2}{3}\right]$
y)	$\lim\limits_{x\to \frac{\pi}{2}} \left(\dfrac{x}{\operatorname{cotg} x} - \dfrac{\pi}{2\cos x}\right)$	$[-1]$	z)	$\lim\limits_{x\to 0} \dfrac{\cosh x - \cos x}{x^2}$	$[1]$

Kapitola 7

Průběh funkce

Příklad 7.1. Určete maximální intervaly ryzí momotonie následujících funkcí:

a) $f(x) = x^2 e^x$ $[\text{rostoucí na } (-\infty; -2\rangle, \langle 0; +\infty), \text{klesající na } \langle -2; 0\rangle]$

b) $f(x) = 12x^5 - 15x^4 - 40x^3 + 60$ $[\text{rostoucí na } (-\infty; -1\rangle, \langle 2; +\infty), \text{klesající na } \langle -1; 0\rangle, \langle 0; 2\rangle]$

c) $f(x) = xe^{-x^2}$ $\left[\text{rostoucí na } \left\langle -\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}} \right\rangle, \text{klesající na } \left(-\infty; -\frac{1}{\sqrt{2}}\right\rangle, \left\langle \frac{1}{\sqrt{2}}; +\infty\right)\right]$

d) $f(x) = \dfrac{x^2}{\ln x}$ $[\text{rostoucí na } \langle \sqrt{e}; +\infty), \text{klesající na } (0; 1) \cup (1; \sqrt{e}\rangle]$

e*) $f(x) = \dfrac{2x}{1 + x^2}$ $[\text{rostoucí na } \langle -1; 1\rangle, \text{klesající na } (-\infty; -1\rangle, \langle 1; +\infty)]$

f) $f(x) = x + |\sin(2x)|$ $\left[\text{rostoucí na } \left\langle \frac{3k-1}{3}\pi; \frac{3k+1}{3}\pi \right\rangle, \text{klesající na } \left\langle \frac{3k+1}{3}\pi; \frac{2k+2}{3}\pi \right\rangle\right]$

g) $f(x) = \dfrac{x^2}{2^x}$ $\left[\text{rostoucí na } \left\langle 0; \frac{2}{\ln 2} \right\rangle, \text{klesající na } (-\infty; 0\rangle, \left\langle \frac{2}{\ln 2}; +\infty\right)\right]$

h) $f(x) = \sqrt[3]{(x^4 - 1)^2}$ [rostoucí na , klesající na]

i) $f(x) = x \ln^2 x$ [rostoucí na , klesající na]

j) $f(x) = x - \dfrac{1}{x}$ $[\text{rostoucí na } (-\infty; 0\rangle, \langle 0; +\infty)]$

k) $f(x) = \ln(1 + x - 4x^2)$ $\left[\text{rostoucí na } \left(\frac{1-\sqrt{17}}{8}; \frac{1}{8} \right\rangle, \text{klesající na } \left\langle \frac{1}{8}; \frac{1+\sqrt{17}}{8}\right)\right]$

Příklad 7.2. Určete lokální exrémy následujících funkcí:

a) $f(x) = x^3 - 6x^2 + 9x - 4$ $\quad$ [lok. min. $x = 3$, lok. max. $x = 1$]

b) $f(x) = \sqrt{x}\ln x$ $\quad$ [lok. min. $x = e^{-2}$]

c) $f(x) = \dfrac{10}{1 + \sin^2 x}$ $\quad$ $\left[\text{lok. min. } x = \dfrac{2k+1}{2}\pi, \text{lok. max. } x = k\pi,\ k \in \mathbb{Z}\right]$

d) $f(x) = \operatorname{arctg} x - \dfrac{1}{2}\ln(1 + x^2)$ $\quad$ [lok. max. $x = 1$]

e) $f(x) = e^x \sin x$ $\quad$ $\left[\text{lok. min. } x = \dfrac{8k-1}{4}\pi, \text{lok. max. } x = \dfrac{8k+3}{4}\pi,\ k \in \mathbb{Z}\right]$

f) $f(x) = |x|e^{-|x-1|}$ $\quad$ [lok. min. $x = 0$, lok. max. $x = 1$]

g) $f(x) = x^3 e^{x^2}$ $\quad$ [lok. extrém neex.]

h) $f(x) = x + 2\operatorname{arccotg} x$ $\quad$ [lok.]

i) $f(x) = \dfrac{2x}{x^2 + 1}$ $\quad$ [lok. min. $x = -1$, lok. max. $x = 1$]

j) $f(x) = \dfrac{\ln(x-1)}{x-1}$ $\quad$ [lok. max. $x = e + 1$]

k) $f(x) = \dfrac{1 - x^3}{x^2}$ $\quad$ [lok. min. $x = \sqrt[3]{-2}$]

Příklad 7.3. Určete maximální intervaly ryzí monotonie a lokální extrémy funkce f (pokud existují):

a) $f(x) = 2x^2 - 5x + 1$ $\quad$ $\left[\text{rostoucí na } \left\langle \dfrac{5}{4}; +\infty\right), \text{klesající na } \left(-\infty; \dfrac{5}{4}\right\rangle, \dfrac{5}{4}\right]$

b*) $f(x) = \dfrac{\ln^2 x}{x}$ $\quad$ [rostoucí na $\langle 1; e^2\rangle$, klesající na $(0; 1\rangle$, $\langle e^2; +\infty)$, e^2]

c) $f(x) = \dfrac{x-1}{x}$ $\quad$ [rostoucí na $(-\infty; 0) \cup (0; +\infty)$]

d) $f(x) = \sqrt[3]{(x^4 - 1)^2}$ $\quad$ [rostoucí na $\langle -1; 0\rangle$, $\langle 1; +\infty)$, klesající na $(-\infty; -1\rangle$, $\langle 0, 1\rangle$, $0, 1$]

e) $f(x) = 3 - 2\sqrt[3]{x^2}$ $\quad$ [rostoucí na $(-\infty; 0\rangle$, klesající na $\langle 0; +\infty)$, 0]

f*) $f(x) = xe^{\frac{1}{x}}$ $\quad$ [rostoucí na $(-\infty; 0)$, $\langle 1; +\infty)$, klesající na $(0; 1\rangle$, 1]

g*) $f(x) = \sqrt{8x - x^2}$ $\quad$ [rostoucí na $\langle 0; 4\rangle$, klesající na $\langle 4; 8\rangle$, 4]

h*) $f(x) = e^{x^3 - 12x}$ $\quad$ [rostoucí na $(-\infty; -2\rangle$, $\langle 2; +\infty)$, klesající na $\langle -2; 2\rangle$, $-2, 2$]

i) $f(x) = 2x + 9\sqrt[3]{(1-x)^2}$ $\quad$ [rostoucí na $(-\infty; -26\rangle$, $\langle 1; +\infty)$, klesající na $\langle -26; 1\rangle$, $-26, 1$]

Příklad 7.4. Najděte maximální intervaly ryzí konvexnosti, konkávnosti a inflexní body funkcí:

a) $f(x) = x^3 + 3x$ $\quad$ $[\text{konvexní na } \langle 0; +\infty), \text{konkávní na } (-\infty; 0\rangle, 0]$

b) $f(x) = \ln(x+2)$ $\quad$ $[\text{konkávní na } (-2; +\infty)]$

c) $f(x) = \dfrac{x^4}{x^3 - 1}$ $\quad$ $\left[\text{konvexní na } (-\infty; -\sqrt[3]{2}\rangle, (1; +\infty), \text{konkávní na } \langle -\sqrt[3]{2}; 1), -\sqrt[3]{2}\right]$

d) $f(x) = x^2 - 1 + \sqrt[3]{x^2}$ $\quad$ $\left[\text{konvexní na } \left(-\infty; -\dfrac{1}{3\sqrt{3}}\right\rangle, \left\langle \dfrac{1}{3\sqrt{3}}; +\infty\right),\right.$

$\left.\text{konkávní na } \left\langle -\dfrac{1}{3\sqrt{3}}; 0\right\rangle, \left\langle 0; \dfrac{1}{3\sqrt{3}}\right\rangle, \pm\dfrac{1}{3\sqrt{3}}\right]$

e) $f(x) = \dfrac{x}{1+x^2}$ $\quad$ $\left[\text{konvexní na } \langle -\sqrt{3}; 0\rangle, \langle \sqrt{3}; +\infty),\right.$

$\left.\text{konkávní na } (-\infty; -\sqrt{3}\rangle, \langle 0; \sqrt{3}\rangle, -\sqrt{3}, 0, \sqrt{3}\right]$

f) $f(x) = \dfrac{\cos x}{2 + \sin x}$ $\quad$ $\left[\text{konvexní na } \left\langle \dfrac{4k+1}{2}\pi; \dfrac{4k+3}{2}\pi\right\rangle,\right.$

$\left.\text{konkávní na } \left\langle 2k\pi; \dfrac{4k+1}{2}\pi\right\rangle, \left\langle \dfrac{4k+3}{2}\pi; (2k+2)\pi\right\rangle, \dfrac{4k+1}{2}\pi,\ k \in \mathbb{Z}\right]$

g) $f(x) = x - \ln(x^2 - 9)$ $\quad$ $[\text{konvexní na } (-\infty; -3), (3; +\infty)]$

h) $f(x) = e^{2x - x^2}$ $\quad$ $\left[\text{konvexní na } \left(-\infty; 1 - \dfrac{1}{2}\sqrt{2}\right\rangle, \left\langle 1 + \dfrac{1}{2}\sqrt{2}; +\infty\right),\right.$

$\left.\text{konkávní na } \left\langle 1 - \dfrac{1}{2}\sqrt{2}; 1 + \dfrac{1}{2}\sqrt{2}\right\rangle, 1 \pm \dfrac{1}{2}\sqrt{2}\right]$

i) $f(x) = \left(\dfrac{1+x}{1-x}\right)^4$ $\quad$ $[\text{konvexní na } \langle -4; 1), (1; +\infty), \text{konkávní na } (-\infty; -4\rangle, -4]$

j) $f(x) = 3x^2 - x^3$ $\quad$ $[\text{konvexní na } (-\infty; 1\rangle, \text{konkávní na } \langle 1; +\infty), 1]$

k) $f(x) = x + x^{5/3}$ $\quad$ $[\text{konvexní na } (-\infty; 0\rangle, \text{konkávní na } \langle 0; +\infty), 0]$

l) $f(x) = \sqrt{1 + x^2}$ $\quad$ $[\text{konvexní na } (-\infty; +\infty)]$

m) $f(x) = x + \sin x$ $\quad$ $[\text{konvexní na } \langle (2k-1)\pi; 2k\pi\rangle, \text{konkávní na } \langle 2k\pi; (2k+1)\pi\rangle, k\pi,\ k \in \mathbb{Z}]$

n) $f(x) = x^x$ $\quad$ $[\text{konvexní na } (0; +\infty)]$

Příklad 7.5. Vyšetřete průběh funkcí:

a) $f(x) = x^3 - 3x + 2$

$\big[D_f = I\!R$, rostoucí na $(-\infty, -1\rangle, \langle 1, +\infty)$, klesající na $\langle -1; 1\rangle$, lok. min. $x = 1$, lok. max. $x = -1$, konvexní na $\langle 0; +\infty), $ konkávní na $(-\infty; 0\rangle\big]$

b) $f(x) = \dfrac{x}{x^2+1}$

$\big[D_f = I\!R$, rostoucí na $\langle -1; 1\rangle$, klesající na $(-\infty; -1\rangle, \langle 1; +\infty)$, lok. min. $x = -1$, lok. max. $x = 1$, konvexní na $\langle -\sqrt{3}; 0\rangle, \langle\sqrt{3}; +\infty),$ konkávní na $(-\infty; -\sqrt{3}\rangle, \langle 0; \sqrt{3}\rangle$, asymptoty $y = 0\big]$

c) $f(x) = \dfrac{x}{3-x^2}$

$\big[D_f = I\!R - \{-\sqrt{3}; \sqrt{3}\}$, rostoucí na $(-\infty; -\sqrt{3}), (-\sqrt{3}; \sqrt{3}), (\sqrt{3}; +\infty)$, konvexní na $(-\infty; -\sqrt{3}), \langle 0; \sqrt{3}),$ konkávní na $(-\sqrt{3}; 0\rangle, (\sqrt{3}; +\infty)$, asymptoty $x = \pm\sqrt{3}, y = 0\big]$

d) $f(x) = \ln(4 - x^2)$

$\big[D_f = (-2; 2)$, rostoucí na $(-2; 0\rangle$, klesající na $\langle 0; 2)$, lok. max. $x = 0$, konkávní na $(-2; 2)$, asymptoty $x = \pm 2\big]$

e) $f(x) = \ln\left(\dfrac{x+1}{1-x}\right)$

$\big[D_f = (-1; 1)$, rostoucí na $(-1; 1)$, konkávní na $(-1; 0\rangle$, konvexní na $\langle 0; 1)$, asymptoty $x = \pm 1\big]$

f) $f(x) = \sqrt[3]{2x^2 - x^3}$ []

g) $f(x) = xe^{\frac{1}{x}}$ []

h) $f(x) = xe^{-\frac{x^2}{2}}$ []

i) $f(x) = \arcsin\left(\dfrac{2x}{1+x^2}\right)$ []

j) $f(x) = (x+1)(x-2)^2$

$\big[D_f = I\!R$, rostoucí na $(-\infty; 0\rangle, \langle 2; +\infty)$, klesající na $\langle 0; 2\rangle$, lok. min. $x = 2$, lok. max. $x = 0$, konvexní na $\langle 1; +\infty),$ konkávní na $(-\infty; 1\rangle\big]$

k) $f(x) = \dfrac{x^2(x-1)}{(x+1)^2}$ []

l) $f(x) = \sqrt{8x^2 - x^4}$ []

m) $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$ []

n) $f(x) = \cos x - \dfrac{1}{2}\cos(2x)$ []

o) $f(x) = \dfrac{\sin x}{2 + \cos x}$ []

p) $f(x) = (1 + x^2)e^{-x^2}$ []

q) $f(x) = \ln(x + \sqrt{1 + x^2})$ []

r) $f(x) = \sqrt[3]{1 - x^2}$ []

s) $f(x) = x + \dfrac{\ln x}{x}$

$\big[D_f = (0; +\infty)$, rostoucí na $(0; +\infty)$, konvexní na $\langle e^{\frac{3}{2}}; +\infty),$ konkávní na $(0; e^{\frac{3}{2}}\rangle$, asymptoty $x = 0,\ y = x\big]$

t) $f(x) = \dfrac{x^4}{(x+1)^3}$ []

u) $f(x) = \dfrac{3x^2 + 4x + 4}{x^2 + x + 1}$ []

v) $f(x) = \dfrac{x}{x^2+1}$ []

w) $f(x) = 3 - \sqrt[3]{x-2}$ []

x) $f(x) = (x^2-1)e^x$ []

y) $f(x) = \dfrac{x^2}{x^2-1}$

$\big[D_f = \mathbb{R} - \{-1;1\}$, rostoucí na $(-\infty;-1),(-1;0\rangle$, klesající na $\langle 0;1),(1;\infty)$, lok. max. $x=0$, konkávní na $(-1;1)$, konvexní na $(-\infty;-1),(1;\infty)$, asymptoty $x=\pm 1$, $y=1\big]$

z) $f(x) = \dfrac{x^3}{3-x^2}$

$\big[D_f = \mathbb{R} - \left\{-\sqrt{3};\sqrt{3}\right\}$, rostoucí na $\langle -3;-\sqrt{3}),(-\sqrt{3};\sqrt{3}),(\sqrt{3};3\rangle$, klesající na $(-\infty;-3\rangle,\langle 3;\infty)$, lok. max. $x=3$, lok. min. $x=-3$, konkávní na $(-\sqrt{3};0\rangle,(\sqrt{3};\infty)$, konvexní na $(-\infty;-\sqrt{3}),\langle 0,\sqrt{3})$, asymptoty $x=\pm\sqrt{3}$, $y=-x\big]$

Kapitola 8

Primitivní funkce

Příklad 8.1. Integrujte rozkladem a přímou metodou dané funkce:

a*)	$\displaystyle\int \left(x^3 - \frac{1}{x} + \frac{\sqrt[4]{x}}{2}\right) \mathrm{d}x$	$\left[\dfrac{x^4}{4} - \ln|x| + \dfrac{2}{5}x\sqrt[4]{x}\right]$
b)	$\displaystyle\int \frac{(x+2)^3}{x^3} \mathrm{d}x$	$\left[x + 6\ln|x| - \dfrac{12}{x} - \dfrac{4}{x^2}\right]$
c)	$\displaystyle\int \frac{x^5 + 2x^4 - x^2}{x^3} \mathrm{d}x$	$\left[\dfrac{x^3}{3} + x^2 - \ln|x|\right]$
d*)	$\displaystyle\int \left(\frac{3}{x^4} + \frac{1}{\sqrt{x}}\right) \mathrm{d}x$	$\left[-\dfrac{1}{x^3} + 2\sqrt{x}\right]$
e)	$\displaystyle\int \left(3\sqrt[5]{x} - 7x\right) \mathrm{d}x$	$\left[\dfrac{5x^{6/5} - 7x^2}{2}\right]$
f)	$\displaystyle\int \frac{x^3 - 2x + 1}{x^3} \mathrm{d}x$	$\left[x - \dfrac{1}{2x^2} + \dfrac{2}{x}\right]$
g)	$\displaystyle\int \left(\frac{1-x}{x}\right)^2 \mathrm{d}x$	$\left[x - 2\ln|x| - \dfrac{1}{x}\right]$
h)	$\displaystyle\int \left(x + \frac{1}{x} + \sqrt{x} + \frac{1}{\sqrt{x}}\right) \mathrm{d}x$	$\left[\dfrac{x^2}{2} + \ln|x| + \dfrac{2x\sqrt{x}}{3} + 2\sqrt{x}\right]$
i)	$\displaystyle\int \left(\frac{14\sqrt{x^3}}{3} - \frac{11}{x^{5/3}} - \frac{4}{3x^2}\right) \mathrm{d}x$	$\left[\dfrac{28x^{5/2}}{15} + \dfrac{33}{2x^{2/3}} + \dfrac{4}{3u}\right]$
j)	$\displaystyle\int (x^3 - 3x^2 + 4x - 7) \,\mathrm{d}x$	$\left[\dfrac{x^4}{4} - x^3 + 2x^2 - 7x\right]$
k)	$\displaystyle\int \left(\frac{4}{\sqrt{3x}} + (3-2x)^2\right) \mathrm{d}x$	$\left[\dfrac{8}{3}\sqrt{3x} + 9x - 6x^2 + \dfrac{4}{3}x^3\right]$
l)	$\displaystyle\int x(2x-5) \,\mathrm{d}x$	$\left[\dfrac{2}{3}x^3 - \dfrac{5}{2}x^2\right]$
m)	$\displaystyle\int \frac{x^4 - 10x^2 + 5}{x^2} \mathrm{d}x$	$\left[\dfrac{x^3}{3} - 10x - \dfrac{5}{x}\right]$
n)	$\displaystyle\int \left(\sqrt{2x} + \sqrt{\frac{2}{x}}\right) \mathrm{d}x$	$\left[2\sqrt{2x}\left(\dfrac{x}{3} + 1\right)\right]$
o)	$\displaystyle\int \frac{4x - 2\sqrt{x}}{x} \mathrm{d}x$	$\left[4(x - \sqrt{x})\right]$
p)	$\displaystyle\int (\sqrt{x}+1)(x - \sqrt{x} + 1) \,\mathrm{d}x$	$\left[\dfrac{2}{5}x^{5/2} + x\right]$
q)	$\displaystyle\int (\sqrt{x}+1)^2 \,\mathrm{d}x$	$\left[x + \dfrac{4}{3}x\sqrt{x} + \dfrac{x^2}{2}\right]$

Příklad 8.2. Integrujte rozkladem a přímou metodou dané funkce:

a) $\displaystyle\int \left(1 - \frac{1}{\sqrt[3]{x}}\right)^2 \mathrm{d}x$ $\qquad \left[x - 3x^{2/3} + 3x^{1/3}\right]$

b) $\displaystyle\int \sqrt{x}(1 - x^2)\,\mathrm{d}x$ $\qquad \left[\frac{2}{3}x^{3/2} - \frac{2}{3}x^{7/2}\right]$

c) $\displaystyle\int \frac{\sqrt{x^4 + 2 + x^{-4}}}{x^3}\,\mathrm{d}x$ $\qquad \left[\ln|x| - \frac{1}{4x^4}\right]$

d) $\displaystyle\int \frac{2 - x^2}{x + \sqrt{2}}\,\mathrm{d}x$ $\qquad \left[\sqrt{2}x - \frac{x^2}{2}\right]$

e) $\displaystyle\int \left(\frac{(2\sqrt{x} + 1)^2}{x^2} + \cos^{-2} x\right)\mathrm{d}x$ $\qquad \left[4\ln|x| - \frac{8}{\sqrt{x}} - \frac{1}{x} + \operatorname{tg} x\right]$

f) $\displaystyle\int \left(\sin x - \frac{1}{\cos^2 x}\right)\mathrm{d}x$ $\qquad [-\cos x - \operatorname{tg} x]$

g) $\displaystyle\int \frac{5\sin^2 x + 3\cos^2 x}{2\sin^2 x \cos^2 x}\,\mathrm{d}x$ $\qquad \left[\frac{5\operatorname{tg} x - 3\operatorname{cotg} x}{2}\right]$

h) $\displaystyle\int \frac{1}{2}\sqrt{e^x}\,\mathrm{d}x$ $\qquad \left[\sqrt{e^x}\right]$

i*) $\displaystyle\int \frac{3 - 2\operatorname{cotg}^2 x}{\cos^2 x}\,\mathrm{d}x$ $\qquad [3\operatorname{tg} x + 2\operatorname{cotg} x]$

j) $\displaystyle\int \frac{1}{\sin^2 x \cos^2 x}\,\mathrm{d}x$ $\qquad [\operatorname{tg} x - \operatorname{cotg} x]$

k) $\displaystyle\int e^x\left(1 + \frac{e^{-x}}{\cos^2 x}\right)\mathrm{d}x$ $\qquad [e^x + \operatorname{tg} x]$

l) $\displaystyle\int \frac{4}{\sqrt{4 - 4x^2}}\,\mathrm{d}x$ $\qquad [2\arcsin x]$

m) $\displaystyle\int e^x\left(1 + \frac{e^x}{3}\right)\mathrm{d}x$ $\qquad \left[e^x + \frac{e^{2x}}{6}\right]$

n) $\displaystyle\int \frac{3 + e^{-x}\sin x}{e^{-x}}\,\mathrm{d}x$ $\qquad [3e^x - \cos x]$

o) $\displaystyle\int \frac{e^{2x} - 1}{e^x}\,\mathrm{d}x$ $\qquad [e^x + e^{-x}]$

p) $\displaystyle\int \frac{-4}{\sqrt{16 - 16x^2}}\,\mathrm{d}x$ $\qquad [\arccos x]$

q) $\displaystyle\int \frac{\sqrt{1 + x^2}}{\sqrt{1 - x^4}}\,\mathrm{d}x$ $\qquad [\arcsin x]$

Příklad 8.3. Integrujte metodou per partes dané funkce:

a) $\displaystyle\int x \operatorname{arctg} x \,\mathrm{d}x$ $\left[\dfrac{(x^2+1)}{2}\operatorname{arctg} x - \dfrac{x}{2}\right]$

b*) $\displaystyle\int x \cos x \,\mathrm{d}x$ $[x \sin x + \cos x]$

c) $\displaystyle\int x \sin x \,\mathrm{d}x$ $[\sin x - x \cos x]$

d) $\displaystyle\int (3x+2) \cos x \,\mathrm{d}x$ $[3 \cos x + (3x+2) \sin x]$

e*) $\displaystyle\int x^3 e^x \,\mathrm{d}x$ $[x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x]$

f) $\displaystyle\int x^2 \cos x \,\mathrm{d}x$ $[x^2 \sin x - 2 \sin x + 2x \cos x]$

g) $\displaystyle\int x^2 \cos^2 x \,\mathrm{d}x$ $\left[\dfrac{x^3}{6} + \left(\dfrac{x^2}{4} - \dfrac{1}{8}\right)\sin(2x) + \dfrac{x}{4}\cos^2 x\right]$

h) $\displaystyle\int (x^2 - 3x + 2) e^x \,\mathrm{d}x$ $[e^x(x^2 - 5x + 7)]$

i) $\displaystyle\int x^2 \ln x \,\mathrm{d}x$ $\left[\dfrac{1}{3}x^3 \ln x - \dfrac{x^3}{9}\right]$

j) $\displaystyle\int x^3 \operatorname{arctg} x \,\mathrm{d}x$ $\left[\dfrac{\operatorname{arctg} x}{4}(x^4 - 1) - \dfrac{x^3 - 3x}{12}\right]$

k) $\displaystyle\int \operatorname{arctg} x \,\mathrm{d}x$ $\left[x \operatorname{arctg} x - \dfrac{1}{2}\ln(x^2+1)\right]$

l) $\displaystyle\int x \arcsin x \,\mathrm{d}x$ $\left[\dfrac{x^2 \arcsin x}{2} + \dfrac{x\sqrt{1-x^2}}{4} - \dfrac{\arcsin x}{4}\right]$

m) $\displaystyle\int \frac{\arcsin x}{\sqrt{1-x^2}} \,\mathrm{d}x$ $\left[\dfrac{1}{2}\arcsin^2 x\right]$

n) $\displaystyle\int e^x \cos x \,\mathrm{d}x$ $\left[\dfrac{1}{2}e^x(\cos x + \sin x)\right]$

o) $\displaystyle\int e^x \sin x \,\mathrm{d}x$ $\left[\dfrac{1}{2}e^x(\sin x - \cos x)\right]$

p) $\displaystyle\int e^x \sin^2 x \,\mathrm{d}x$ $\left[\dfrac{1}{5}e^x \sin x(\sin x - 2\cos x) + \dfrac{2}{5}e^x\right]$

q) $\displaystyle\int x e^x \,\mathrm{d}x$ $[e^x(x-1)]$

r) $\displaystyle\int \ln x \,\mathrm{d}x$ $[x \ln x - x]$

s) $\displaystyle\int x \ln x \,\mathrm{d}x$ $\left[\dfrac{x^2}{2}\left(\ln x - \dfrac{1}{2}\right)\right]$

t) $\displaystyle\int \cos^2 x \,\mathrm{d}x$ $\left[\dfrac{1}{2}(x + \sin x \cos x)\right]$

u) $\displaystyle\int \sin^2 x \,\mathrm{d}x$ $\left[\dfrac{1}{2}(x - \sin x \cos x)\right]$

Příklad 8.4. Integrujte pomocí substituční metody dané funkce:

a) $\displaystyle\int \sin^3 x \cos x \,\mathrm{d}x$ $\quad\left[\frac{1}{4}\sin^4 x\right]$

b) $\displaystyle\int 6x\sin(3x^2)\,\mathrm{d}x$ $\quad\left[-\cos(3x^2)\right]$

c) $\displaystyle\int \frac{4\mathrm{tg}^3 x}{\cos^2 x}\,\mathrm{d}x$ $\quad\left[\mathrm{tg}^4 x\right]$

d) $\displaystyle\int \cos x\cdot\sqrt{\sin x}\,\mathrm{d}x$ $\quad\left[\frac{2(\sin x)^{3/2}}{3}\right]$

e) $\displaystyle\int -4xe^{-2x^2}\,\mathrm{d}x$ $\quad\left[e^{-2x^2}\right]$

f) $\displaystyle\int 6x^2e^{-2x^3}\,\mathrm{d}x$ $\quad\left[-e^{-2x^3}\right]$

g) $\displaystyle\int 2e^{2\sin x}\cos x\,\mathrm{d}x$ $\quad\left[e^{2\sin x}\right]$

h) $\displaystyle\int \frac{3\ln^2 x}{x}\,\mathrm{d}x$ $\quad\left[\ln^3 x\right]$

i) $\displaystyle\int \frac{3\sqrt{\ln x}}{x}\,\mathrm{d}x$ $\quad\left[2\ln^{3/2} x\right]$

j) $\displaystyle\int \frac{2x}{(1+x^2)^2}\,\mathrm{d}x$ $\quad\left[\frac{-1}{1+x^2}\right]$

k) $\displaystyle\int \frac{8x^2}{\sqrt[3]{(8x^3+27)^2}}\,\mathrm{d}x$ $\quad\left[\sqrt[3]{8x^3+27}\right]$

l) $\displaystyle\int \frac{5x^4}{2\sqrt{4+x^5}}\,\mathrm{d}x$ $\quad\left[\sqrt{4+x^5}\right]$

m) $\displaystyle\int \frac{6x-5}{2\sqrt{3x^2-5x+6}}\,\mathrm{d}x$ $\quad\left[\sqrt{3x^2-5x+6}\right]$

n) $\displaystyle\int \frac{3\cos x}{\sin^4 x}\,\mathrm{d}x$ $\quad\left[\frac{-1}{\sin^3 x}\right]$

o) $\displaystyle\int \frac{\sin x}{2\sqrt{\cos^3 x}}\,\mathrm{d}x$ $\quad\left[\frac{1}{\sqrt{\cos x}}\right]$

p) $\displaystyle\int \frac{4\cos x}{\sqrt[3]{1+2\sin x}}\,\mathrm{d}x$ $\quad\left[3(1+2\sin x)^{2/3}\right]$

q) $\displaystyle\int \frac{\sin(2x)}{2\sqrt{1+\cos^2 x}}\,\mathrm{d}x$ $\quad\left[-\sqrt{1+\cos^2 x}\right]$

r) $\displaystyle\int \frac{1}{x\ln x\ln(\ln x)}\,\mathrm{d}x$ $\quad\left[\ln|\ln(\ln x)|\right]$

s) $\displaystyle\int \frac{6x}{\sqrt{4-9x^4}}\,\mathrm{d}x$ $\quad\left[\arcsin\frac{3x^2}{2}\right]$

t) $\displaystyle\int \frac{2e^x}{\sqrt{2-4e^{2x}}}\,\mathrm{d}x$ $\quad\left[\arcsin(\sqrt{2}e^x)\right]$

u) $\displaystyle\int 6\,\mathrm{tg}\,(3x)\,\mathrm{d}x$ $\quad\left[-2\ln|\cos(3x)|\right]$

Příklad 8.5. Integrujte pomocí substituční metody dané funkce:

a) $\displaystyle\int \frac{1}{x\sqrt{1-\ln^2 x}}\,\mathrm{d}x$ $\quad [\arcsin(\ln x)]$

b) $\displaystyle\int \frac{30x}{3x^4+5}\,\mathrm{d}x$ $\quad \left[\sqrt{15}\,\mathrm{arctg}\,\frac{x^2\sqrt{15}}{5}\right]$

c) $\displaystyle\int \frac{18x}{9+(3x^2+1)^2}\,\mathrm{d}x$ $\quad \left[\mathrm{arctg}\left(x^2+\frac{1}{3}\right)\right]$

d) $\displaystyle\int \frac{-2}{\mathrm{tg}\,x\sin^2 x}\,\mathrm{d}x$ $\quad \left[\frac{1}{\sin^2 x}\right]$

e) $\displaystyle\int 4\sin x\cos^3 x\,\mathrm{d}x$ $\quad [-\cos^4 x]$

f) $\displaystyle\int \frac{2\ln x}{x}\,\mathrm{d}x$ $\quad [\ln^2 x]$

g) $\displaystyle\int \frac{2\,\mathrm{arctg}\,x}{1+x^2}\,\mathrm{d}x$ $\quad [\mathrm{arctg}^2 x]$

h) $\displaystyle\int \sqrt{1+2x}\,\mathrm{d}x$ $\quad \left[\frac{(1+2x)^{3/2}}{3}\right]$

i) $\displaystyle\int \frac{1}{\sqrt{5-4x}}\,\mathrm{d}x$ $\quad \left[-\frac{\sqrt{5-4x}}{2}\right]$

j) $\displaystyle\int \frac{3x}{(x^2+1)^2}\,\mathrm{d}x$ $\quad \left[-\frac{3}{2(x^2+1)}\right]$

k) $\displaystyle\int x\sqrt{2x^2+7}\,\mathrm{d}x$ $\quad \left[\frac{(2x^2+7)^{3/2}}{6}\right]$

l) $\displaystyle\int 9x^2\sqrt[3]{x^3+10}\,\mathrm{d}x$ $\quad \left[\frac{9(x^3+10)^{4/3}}{4}\right]$

m) $\displaystyle\int \frac{4x}{\sqrt[3]{8-x^2}}\,\mathrm{d}x$ $\quad \left[-3(8-x^2)^{2/3}\right]$

n) $\displaystyle\int \frac{7}{(1+2x)^3}\,\mathrm{d}x$ $\quad \left[\frac{-7}{4(1+2x)^2}\right]$

o) $\displaystyle\int 3\cos^4 x\sin x\,\mathrm{d}x$ $\quad \left[-\frac{3}{5}\cos^5 x\right]$

p) $\displaystyle\int \frac{x}{x^2-1}\,\mathrm{d}x$ $\quad \left[\frac{1}{2}\ln|x^2-1|\right]$

q) $\displaystyle\int \frac{1}{\cos^2(1-x)}\,\mathrm{d}x$ $\quad [\mathrm{tg}\,(x-1)]$

r) $\displaystyle\int \frac{\cos x}{3\sin^{2/3} x}\,\mathrm{d}x$ $\quad \left[\sqrt[3]{\sin x}\right]$

s) $\displaystyle\int \frac{1}{\sqrt{1-x^2}\arcsin x}\,\mathrm{d}x$ $\quad [\ln|\arcsin x|]$

t) $\displaystyle\int \frac{x-\mathrm{arctg}\,x}{1+x^2}\,\mathrm{d}x$ $\quad \left[\frac{1}{2}\left(-\mathrm{arctg}^2 x+\ln(1+x^2)\right)\right]$

u) $\displaystyle\int \frac{1}{x(1+\ln^2 x)}\,\mathrm{d}x$ $\quad [\mathrm{arctg}\,(\ln x)]$

Příklad 8.6. Integrujte pomocí substituční metody dané funkce:

a) $\displaystyle\int (4x-3)^4\,\mathrm{d}x$ $\qquad\left[\dfrac{(4x-3)^5}{20}\right]$

b) $\displaystyle\int (2x+1)^3\,\mathrm{d}x$ $\qquad\left[\dfrac{(2x+1)^4}{8}\right]$

c) $\displaystyle\int \frac{1}{6}\left(1-\frac{x}{6}\right)^{-2}\,\mathrm{d}x$ $\qquad\left[\dfrac{6}{6-x}\right]$

d) $\displaystyle\int \frac{12}{(3x-7)^5}\,\mathrm{d}x$ $\qquad\left[-\dfrac{1}{(3x-7)^4}\right]$

e) $\displaystyle\int \frac{1}{x^2-6x+9}\,\mathrm{d}x$ $\qquad\left[-\dfrac{1}{x-3}\right]$

f) $\displaystyle\int 33(8-3x)^{6/5}\,\mathrm{d}x$ $\qquad\left[-5(8-3x)^{11/5}\right]$

g) $\displaystyle\int \sqrt[3]{5-6x}\,\mathrm{d}x$ $\qquad\left[-\dfrac{(5-6x)^{4/3}}{8}\right]$

h) $\displaystyle\int \frac{1}{\sqrt{4x+9}}\,\mathrm{d}x$ $\qquad\left[\dfrac{\sqrt{4x+9}}{2}\right]$

i) $\displaystyle\int \frac{1}{\sqrt{3-2x}}\,\mathrm{d}x$ $\qquad\left[-\sqrt{3-2x}\right]$

j) $\displaystyle\int \sin(2x-5)\,\mathrm{d}x$ $\qquad\left[-\dfrac{1}{2}\cos(2x-5)\right]$

k) $\displaystyle\int \frac{1}{\sin^2(3x-7)}\,\mathrm{d}x$ $\qquad\left[-\dfrac{1}{3}\operatorname{cotg}(3x-7)\right]$

l) $\displaystyle\int \frac{1}{\cos^2(8x)}\,\mathrm{d}x$ $\qquad\left[\dfrac{1}{8}\operatorname{tg}(8t)\right]$

m) $\displaystyle\int \frac{4}{1-\cos(4x)}\,\mathrm{d}x$ $\qquad\left[-\operatorname{cotg}(2x)\right]$

n) $\displaystyle\int 14e^{7x-8}\,\mathrm{d}x$ $\qquad\left[2e^{7x-8}\right]$

o) $\displaystyle\int 3e^{-3x+1}\,\mathrm{d}x$ $\qquad\left[-e^{-3x+1}\right]$

p) $\displaystyle\int \frac{e^{2x}-1}{e^x}\,\mathrm{d}x$ $\qquad\left[e^x+\dfrac{1}{e^x}\right]$

q) $\displaystyle\int \frac{e^{x/2}-e^{-x/2}}{2}\,\mathrm{d}x$ $\qquad\left[e^{x/2}+e^{-x/2}\right]$

Příklad 8.7. Integrujte pomocí substituční metody dané funkce:

a) $\displaystyle\int \frac{1}{x^2+4x+5}\,\mathrm{d}x$ $\quad[\operatorname{arctg}(x+2)]$

b) $\displaystyle\int \frac{3}{x^2+3x+3}\,\mathrm{d}x$ $\quad\left[2\sqrt{3}\operatorname{arctg}\frac{\sqrt{3}(2x+3)}{3}\right]$

c) $\displaystyle\int \frac{2}{1-3x+3x^2-x^3}\,\mathrm{d}x$ $\quad\left[\frac{1}{(1-x)^2}\right]$

d) $\displaystyle\int \frac{2}{x^2-2x+5}\,\mathrm{d}x$ $\quad\left[\operatorname{arctg}\frac{x-1}{2}\right]$

e) $\displaystyle\int \frac{1}{\sqrt{1-(2x+3)^2}}\,\mathrm{d}x$ $\quad\left[\frac{1}{2}\arcsin(2x+3)\right]$

f) $\displaystyle\int \frac{50}{\sqrt{1-(25x)^2}}\,\mathrm{d}x$ $\quad[2\arcsin(25x)]$

g) $\displaystyle\int \frac{2}{\sqrt{3+2x-x^2}}\,\mathrm{d}x$ $\quad\left[2\arcsin\frac{x-1}{2}\right]$

h) $\displaystyle\int \frac{1}{\sqrt{-2x-x^2}}\,\mathrm{d}x$ $\quad[\arcsin(x+1)]$

i) $\displaystyle\int \frac{3}{\sqrt{2x-x^2}}\,\mathrm{d}x$ $\quad[3\arcsin(x-1)]$

j) $\displaystyle\int \frac{5}{\sqrt{36-(5x)^2}}\,\mathrm{d}x$ $\quad\left[\arcsin\frac{5x}{6}\right]$

k) $\displaystyle\int \frac{1}{1+(x+1)^2}\,\mathrm{d}x$ $\quad[\operatorname{arctg}(x+1)]$

l) $\displaystyle\int \frac{2}{x^2-2x+5}\,\mathrm{d}x$ $\quad\left[\operatorname{arctg}\frac{x-1}{2}\right]$

m) $\displaystyle\int \frac{5}{1+(2-5x)^2}\,\mathrm{d}x$ $\quad[\operatorname{arctg}(5x-2)]$

Příklad 8.8. Integrujte pomocí rozkladu na parciální zlomky dané funkce:

a) $\displaystyle\int \frac{x}{(x+1)(2x+1)}\,\mathrm{d}x$ $\quad\displaystyle\left[\ln|x+1| - \frac{1}{2}\ln|2x+1|\right]$

b) $\displaystyle\int \frac{x}{2x^2-3x-2}\,\mathrm{d}x$ $\quad\displaystyle\left[\frac{2}{5}\ln|x-2| + \frac{1}{10}\ln|2x+1|\right]$

c) $\displaystyle\int \frac{5x-14}{x^3-x^2-4x+4}\,\mathrm{d}x$ $\quad[-\ln|x-2| + 3\ln|x-1| - 2\ln|x+2|]$

d) $\displaystyle\int \frac{33}{6x^3-7x^2-3x}\,\mathrm{d}x$ $\quad[9\ln|3x+1| - 11\ln|x| + 2\ln|2x-3|]$

e) $\displaystyle\int \frac{1}{x(2+x)}\,\mathrm{d}x$ $\quad\displaystyle\left[\frac{1}{2}\ln\left|\frac{x}{2+x}\right|\right]$

f) $\displaystyle\int \frac{12(x-1)}{(x+1)(x^2-4)}\,\mathrm{d}x$ $\quad[\ln|x-2| + 8\ln|x+1| - 9\ln|x+2|]$

g) $\displaystyle\int \frac{32x}{(2x-1)(4x^2-16x+15)}\,\mathrm{d}x$ $\quad[\ln|2x-1| + 5\ln|2x-5| - 6\ln|2x-3|]$

h) $\displaystyle\int \frac{6(x^3+1)}{x^3-5x^2+6x}\,\mathrm{d}x$ $\quad[6x - 27\ln|x-2| + \ln|x| + 56\ln|x-3|]$

i) $\displaystyle\int \frac{1}{6+x-x^2}\,\mathrm{d}x$ $\quad\displaystyle\left[\frac{1}{5}\ln\left|\frac{2+x}{3-x}\right|\right]$

j) $\displaystyle\int \frac{18(3x^2+1)}{x^4-3x^2+2x}\,\mathrm{d}x$ $\quad\displaystyle\left[\frac{-24}{x-1} + \ln\frac{|x|^9\,|x-1|^4}{|x+2|^{1}3}\right]$

k) $\displaystyle\int \frac{x^2-3x+2}{x^3+2x^2+x}\,\mathrm{d}x$ $\quad\displaystyle\left[2\ln|x| - \ln|x+1| + \frac{6}{x+1}\right]$

l) $\displaystyle\int \frac{4(3x^2+1)}{(x^2-1)^3}\,\mathrm{d}x$ $\quad\displaystyle\left[-\frac{1}{(x-1)^2} + \frac{1}{(x+1)^2}\right]$

m) $\displaystyle\int \frac{2(x^2-4x+5)}{x+3}\,\mathrm{d}x$ $\quad[x^2 - 14x + 52\ln|x+3|]$

n) $\displaystyle\int \frac{x}{x^2-x-2}\,\mathrm{d}x$ $\quad\displaystyle\left[\frac{1}{3}\ln|x+1| + \frac{2}{3}\ln|x-2|\right]$

o) $\displaystyle\int \frac{1}{(3+x)(1+2x+x^2)}\,\mathrm{d}x$ $\quad\displaystyle\left[\frac{-1}{2(1+x)} + \frac{1}{4}\ln\left|\frac{3+x}{1+x}\right|\right]$

p) $\displaystyle\int \frac{x}{4-4x+x^2}\,\mathrm{d}x$ $\quad\displaystyle\left[\frac{2}{2-x} + \ln|2-x|\right]$

q) $\displaystyle\int \frac{1}{x(16-24x+9x^2)}\,\mathrm{d}x$ $\quad\displaystyle\left[\frac{1}{16}\left(\frac{4}{4-3x} + \ln\left|\frac{x}{3x-4}\right|\right)\right]$

r) $\displaystyle\int \frac{x^2}{(x+2)^2(x+4)^2}\,\mathrm{d}x$ $\quad\displaystyle\left[\frac{-1}{x+2} + 2\ln\left|\frac{x+4}{x+2}\right| - \frac{4}{x+4}\right]$

s) $\displaystyle\int \frac{2x^2+41x-91}{(x-1)(x^2-x-12)}\,\mathrm{d}x$ $\quad\displaystyle\left[\ln\left|\frac{(x-1)^4(x-4)^5}{(x+3)^7}\right|\right]$

t) $\displaystyle\int \frac{1}{2x^2+9x-5}\,\mathrm{d}x$ $\quad\displaystyle\left[\frac{1}{11}\ln\left|\frac{2x-1}{x+5}\right|\right]$

u) $\displaystyle\int \frac{3x^3-5x^2+8x}{(x^2-2x+1)(x^2-1)}\,\mathrm{d}x$ $\quad\displaystyle\left[-\frac{3}{2(x-1)} - \frac{2}{x-1} + \ln|(x-1)(x+1)^2|\right]$

Příklad 8.9. Integrujte pomocí rozkladu na parciální zlomky dané funkce:

a) $\displaystyle\int \frac{9x-5}{9x^2-6x+1}\,\mathrm{d}x$ $\qquad\left[\frac{2}{3(3x-1)}+\ln|3x-1|\right]$

b) $\displaystyle\int \frac{x^5+x^4+3x^3+x^2-2}{x^4-1}\,\mathrm{d}x$ $\qquad\left[\frac{x^2}{2}+x+\ln(|x^2-1|\sqrt{x^2+1})+\operatorname{arctg}x\right]$

c) $\displaystyle\int \frac{x^4+x^3-2x^2+2x+3}{x^2+x-2}\,\mathrm{d}x$ $\qquad\left[\frac{x^3}{3}+\ln\sqrt[3]{|x-1|^5|x+2|}\right]$

d) $\displaystyle\int \frac{x+3}{(x^2-x+1)^2}\,\mathrm{d}x$ $\qquad\left[\frac{7x-5}{3(x^2-x+1)}+\frac{14}{3\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}\right]$

e) $\displaystyle\int \frac{3x+1}{(x^2+2)^3}\,\mathrm{d}x$ $\qquad\left[\frac{x-6}{8(x^2+2)^2}+\frac{3x}{32(x^2+2)}+\frac{3}{32\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}\right]$

f) $\displaystyle\int \frac{5x+3}{(x^2-2x+5)^3}\,\mathrm{d}x$ $\qquad\left[\frac{2x-7}{4(x^2-2x+5)^2}+\frac{3x-3}{16(x^2-2x+5)}+\frac{3}{32}\operatorname{arctg}\frac{x-1}{2}\right]$

g) $\displaystyle\int \frac{2x+2}{(x-1)(x^2+1)^2}\,\mathrm{d}x$ $\qquad\left[\frac{1}{2}\ln\frac{(x-1)^2}{x^2+1}+\frac{1}{x^2+1}-\operatorname{arctg}x\right]$

h) $\displaystyle\int \frac{x^2}{x^3+5x^2+8x+4}\,\mathrm{d}x$ $\qquad\left[\ln|x+1|+\frac{4}{x+2}\right]$

i) $\displaystyle\int \frac{5x-1}{x^3-3x-2}\,\mathrm{d}x$ $\qquad\left[\ln\left|\frac{x-2}{x+1}\right|-\frac{2}{x+1}\right]$

j) $\displaystyle\int \frac{4x^2}{1-x^4}\,\mathrm{d}x$ $\qquad\left[\ln\left|\frac{x+1}{x-1}\right|-2\operatorname{arctg}x\right]$

k) $\displaystyle\int \frac{10(7x^2+1)}{x^4+4x^2-5}\,\mathrm{d}x$ $\qquad\left[12\sqrt{5}\operatorname{arctg}\frac{x}{\sqrt{5}}+\ln\left|\frac{x-1}{x+1}\right|\right]$

l) $\displaystyle\int \frac{6x}{x^3+1}\,\mathrm{d}x$ $\qquad\left[-2\ln|x+1|+\ln(x^2-x+1)+2\sqrt{3}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}\right]$

m) $\displaystyle\int \frac{x^2}{x^3+5x^2+8x+4}\,\mathrm{d}x$ $\qquad\left[\ln|x+1|+\frac{4}{x+2}\right]$

n) $\displaystyle\int \frac{x^3+1}{x^3-x^2}\,\mathrm{d}x$ $\qquad\left[x-\ln|x|+\frac{1}{x}+2\ln|x-1|\right]$

o) $\displaystyle\int \frac{x^2}{x^2+2}\,\mathrm{d}x$ $\qquad\left[x-\sqrt{2}\operatorname{arctg}\frac{x}{\sqrt{2}}\right]$

p) $\displaystyle\int \frac{3x^4}{x^2+2}\,\mathrm{d}x$ $\qquad\left[x^3-6x+6\sqrt{2}\operatorname{arctg}\frac{x}{\sqrt{2}}\right]$

q) $\displaystyle\int \frac{1}{x^3-x^2}\,\mathrm{d}x$ $\qquad\left[\frac{1}{x}+\ln\left|\frac{x-1}{x}\right|\right]$

r) $\displaystyle\int \frac{2}{x(x^2+1)}\,\mathrm{d}x$ $\qquad\left[2\ln|x|-\ln(x^2+1)\right]$

s) $\displaystyle\int \frac{6x}{x^3-1}\,\mathrm{d}x$ $\qquad\left[2\ln|x-1|-\ln(x^2+x+1)+2\sqrt{3}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}\right]$

t) $\displaystyle\int \frac{6x^2-x+1}{x^3-x}\,\mathrm{d}x$ $\qquad\left[4\ln|x+1|-\ln|x|+3\ln|x-1|\right]$

u) $\displaystyle\int \frac{4}{(x^2+1)(x^2+x)}\,\mathrm{d}x$ $\qquad\left[4\ln|x|-2\ln|x+1|-\ln(x^2+1)-2\operatorname{arctg}x\right]$

Příklad 8.10. Integrujte pomocí rozkladu na parciální zlomky dané funkce:

a) $\displaystyle\int \frac{4}{(x+1)^2(x^2+1)}\,\mathrm{d}x$ $\quad\left[2\ln|x+1| - \frac{2}{x+1} - \ln(x^2+1)\right]$

b) $\displaystyle\int \frac{2(x^3-6)}{x^4+6x^2+8}\,\mathrm{d}x$ $\quad\left[3\operatorname{arctg}\frac{x}{2} - \ln(x^2+2) - 3\sqrt{2}\operatorname{arctg}\frac{x}{\sqrt{2}} + 2\ln(x^2+4)\right]$

c) $\displaystyle\int \frac{6x}{x^2+2x+4}\,\mathrm{d}x$ $\quad\left[3\ln(x^2+2x+4) - 2\sqrt{3}\operatorname{arctg}\frac{x+1}{\sqrt{3}}\right]$

d) $\displaystyle\int \frac{6(3x+8)}{x^2+2x+10}\,\mathrm{d}x$ $\quad\left[9\ln(x^2+2x+10)10\operatorname{arctg}\frac{x+1}{3}\right]$

e) $\displaystyle\int \frac{10}{(5x+4)^3}\,\mathrm{d}x$ $\quad\left[-\frac{1}{(5x+4)^2}\right]$

f) $\displaystyle\int \frac{28(-5x+16)}{2x^2+7}\,\mathrm{d}x$ $\quad\left[32\sqrt{14}\operatorname{arctg}\frac{2x}{\sqrt{14}} - 35\ln(2x^2+7)\right]$

g) $\displaystyle\int \frac{28}{2x^2+4x+6}\,\mathrm{d}x$ $\quad\left[7\sqrt{2}\operatorname{arctg}\frac{x+1}{\sqrt{2}}\right]$

h) $\displaystyle\int \frac{30}{4x^2+4x+16}\,\mathrm{d}x$ $\quad\left[\sqrt{15}\operatorname{arctg}\frac{2x+1}{\sqrt{15}}\right]$

i) $\displaystyle\int \frac{4x}{(x^2-1)(x^2+1)}\,\mathrm{d}x$ $\quad\left[\ln\left|\frac{x^2-1}{x^2+1}\right|\right]$

Příklad 8.11. Integrujte dané funkce:

a) $\displaystyle\int e^x\cos\frac{x}{2}\,\mathrm{d}x$ $\quad\left[\frac{2}{5}e^x\left(\sin\frac{x}{2} + 2\cos\frac{x}{2}\right)\right]$

b) $\displaystyle\int (x+2)^2e^{-x}\,\mathrm{d}x$ $\quad\left[e^{-x}(-x^2-6x-10)\right]$

c) $\displaystyle\int xe^{2x}\,\mathrm{d}x$ $\quad\left[\frac{1}{4}e^{2x}(2x-1)\right]$

d) $\displaystyle\int x^2\sin(3x)\,\mathrm{d}x$ $\quad\left[-\frac{x^2\cos(3x)}{3} + \frac{2x\sin(3x)}{9} + \frac{2\cos(3x)}{27}\right]$

e) $\displaystyle\int \frac{3x}{e^x}\,\mathrm{d}x$ $\quad\left[-e^{-x}(3x+1)\right]$

f) $\displaystyle\int (1-x)e^{-x}\,\mathrm{d}x$ $\quad\left[xe^{-x}\right]$

g) $\displaystyle\int 3x\sin(3x)\,\mathrm{d}x$ $\quad\left[-x\cos(3x) + \frac{1}{3}\sin(3x)\right]$

h) $\displaystyle\int \sin(2x)\cos x\,\mathrm{d}x$ $\quad\left[-\frac{2}{3}\cos^3 x\right]$

i) $\displaystyle\int x\ln(x-1)\,\mathrm{d}x$ $\quad\left[\frac{1}{2}(x^2-1)\ln(x-1) - \frac{x^2}{4} - \frac{x}{2}\right]$

j) $\displaystyle\int \frac{\ln x}{x^2}\,\mathrm{d}x$ $\quad\left[-\frac{\ln x}{x} - \frac{1}{x}\right]$

k) $\displaystyle\int 4\ln(2x)\,\mathrm{d}x$ $\quad\left[4x\ln(2x) - 4x\right]$

l) $\displaystyle\int \frac{\sin\frac{x}{2}}{e^{-x}}\,\mathrm{d}x$ $\quad\left[-\frac{2}{5}e^x\cos\frac{x}{2} + \frac{4}{5}e^x\sin\frac{x}{2}\right]$

m) $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x$ $\quad\left[\frac{1}{2}\ln^2 x\right]$

n) $\displaystyle\int e^{-2x}\sin(3x)\,\mathrm{d}x$ $\quad\left[-\frac{e^{-2x}}{13}(3\cos(3x) + 2\sin(3x))\right]$

o) $\displaystyle\int e^{3x}\cos^2(3x)\,\mathrm{d}x$ $\quad\left[\frac{e^{3x}}{15}\left((\cos(3x) + 2\sin(3x))\cos(3x) + 2\right)\right]$

Kapitola 9

Určitý integrál a jeho aplikace

Příklad 9.1. Vypočítejte určité integrály:

a) $\displaystyle\int_0^{\pi/2} \sin x \,\mathrm{d}x$ $[1]$

b) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \,\mathrm{d}x$ $[2]$

c) $\displaystyle\int_{-1}^{3} (x^3 - 3x^2 + 1) \,\mathrm{d}x$ $[-4]$

d) $\displaystyle\int_0^{\pi} 5\sin(4x) \,\mathrm{d}x$ $[0]$

e) $\displaystyle\int_0^{3} e^{\frac{x}{3}} \,\mathrm{d}x$ $[3e - 3]$

f) $\displaystyle\int_0^{3} \left(\frac{12}{x^2} + 9\right) \mathrm{d}x$ $[\pi]$

g) $\displaystyle\int_{-\pi/2}^{\pi/2} 2\sin^2(2x) \,\mathrm{d}x$ $[\pi]$

h) $\displaystyle\int_0^{4} \frac{x-1}{x+1} \,\mathrm{d}x$ $[4 - 2\ln 5]$

i) $\displaystyle\int_2^{5} \frac{4}{x} \,\mathrm{d}x$ $\left[4\ln\frac{5}{2}\right]$

j) $\displaystyle\int_{1/2}^{2} \frac{1}{x^2} \,\mathrm{d}x$ $\left[\frac{1}{2}\right]$

k) $\displaystyle\int_{-1}^{1} 2x^3 \,\mathrm{d}x$ $[0]$

l) $\displaystyle\int_{-1}^{1} \frac{1}{x-4} \,\mathrm{d}x$ $\left[\ln\frac{3}{5}\right]$

m) $\displaystyle\int_2^{3} \left(\frac{1}{x^2} + x\right) \mathrm{d}x$ $[-3\ln 2 + 2\ln 3]$

n) $\displaystyle\int_{-2}^{2} \frac{6}{8+3x^2} \,\mathrm{d}x$ $\left[\sqrt{6}\,\mathrm{arctg}\,\frac{\sqrt{6}}{2}\right]$

o) $\displaystyle\int_{-\pi}^{\pi} 2\cos x \sin x \,\mathrm{d}x$ $[0]$

p) $\displaystyle\int_{-\pi/4}^{\pi/4} 4\sin^2 x \,\mathrm{d}x$ $[-2 + \pi]$

q) $\displaystyle\int_{-1}^{1} \frac{1}{\cos(2x)} \,\mathrm{d}x$ $[2\,\mathrm{tg}\,1]$

r) $\displaystyle\int_{-1/2}^{1/2} \mathrm{tg}\, x \,\mathrm{d}x$ $[0]$

s) $\displaystyle\int_1^{4} 3\sqrt{x} \,\mathrm{d}x$ $[14]$

t) $\displaystyle\int_1^{10} \frac{6}{9x} \,\mathrm{d}x$ $\left[\frac{2}{3}\ln 10\right]$

u) $\displaystyle\int_{-2}^{2} \frac{x^2}{x^2+1} \,\mathrm{d}x$ $[4 - 2\,\mathrm{arctg}\,2]$

v) $\displaystyle\int_0^{\pi} 2x\sin^2 x \,\mathrm{d}x$ $\left[\frac{1}{2}\pi^2\right]$

w) $\displaystyle\int_0^{1} x^2 e^{-\frac{x}{2}} \,\mathrm{d}x$ $\left[-26e^{-1/2} + 16\right]$

x) $\displaystyle\int_0^{3} x e^{-\frac{x}{2}} \,\mathrm{d}x$ $\left[-10e^{-3/2} + 4\right]$

y) $\displaystyle\int_{-1}^{1} 4x^2 e^{-2x} \,\mathrm{d}x$ $\left[-5e^{-2} + e^{2}\right]$

z) $\displaystyle\int_0^{\pi} x\cos x \,\mathrm{d}x$ $[-2]$

Příklad 9.2. Vypočítejte určité integrály:

a) $\displaystyle\int_0^{\pi} x^2 \sin x \,\mathrm{d}x$ $\quad [\pi^2 - 4]$

b) $\displaystyle\int_0^{2} \operatorname{arctg} x \,\mathrm{d}x$ $\quad \left[2\operatorname{arctg} 2 - \frac{1}{2}\ln 5\right]$

c) $\displaystyle\int_{-1}^{1} 4x \operatorname{arctg}(2x) \,\mathrm{d}x$ $\quad [5\operatorname{arctg} 2 - 2]$

d) $\displaystyle\int_0^{1} 6 \arcsin \frac{x}{2} \,\mathrm{d}x$ $\quad \left[\pi + 6\sqrt{3} - 12\right]$

e) $\displaystyle\int_0^{1} \frac{2\sqrt{x}}{x+1} \,\mathrm{d}x$ $\quad [4 - \pi]$

f) $\displaystyle\int_2^{3} \frac{e^{\frac{1}{x}}}{x^2} \,\mathrm{d}x$ $\quad [-\sqrt[3]{e} + \sqrt{e}]$

g) $\displaystyle\int_0^{3} \frac{3x}{\sqrt{4x+4}} \,\mathrm{d}x$ $\quad [4]$

h) $\displaystyle\int_1^{2} \frac{2(1+\ln x)}{x} \,\mathrm{d}x$ $\quad [\ln^2 2 + 2\ln 2]$

i) $\displaystyle\int_2^{4} \frac{\sqrt{x}}{\sqrt{x}-1} \,\mathrm{d}x$ $\quad [6 - 2\ln(\sqrt{2}-1) - 2\sqrt{2}]$

j) $\displaystyle\int_0^{4} 12\sqrt{x + \frac{1}{4}} \,\mathrm{d}x$ $\quad [17\sqrt{17} - 1]$

k) $\displaystyle\int_0^{\pi} \sin x \sqrt{1+\cos^2 x} \,\mathrm{d}x$ $\quad \left[\sqrt{2} + \ln(1+\sqrt{2})\right]$

l) $\displaystyle\int_1^{2} \frac{1}{x\sqrt{1-\ln^2 x}} \,\mathrm{d}x$ $\quad [\arcsin(\ln 2)]$

m) $\displaystyle\int_0^{4} \frac{1}{1+\sqrt{x}} \,\mathrm{d}x$ $\quad [4 - 2\ln 3]$

n) $\displaystyle\int_0^{\pi} 8\cos^2 x \sin^2 x \,\mathrm{d}x$ $\quad [\pi]$

o) $\displaystyle\int_0^{\pi} 2(1-\cos x)^3 \,\mathrm{d}x$ $\quad [5\pi]$

p) $\displaystyle\int_1^{2} \frac{6}{6x-1} \,\mathrm{d}x$ $\quad \left[\ln \frac{11}{5}\right]$

q) $\displaystyle\int_0^{1} 12\sqrt{x}\sqrt{\frac{4x+1}{4x}} \,\mathrm{d}x$ $\quad \left[5\sqrt{5} - 1\right]$

r) $\displaystyle\int_0^{2\pi} \sqrt{1-\cos x} \,\mathrm{d}x$ $\quad \left[4\sqrt{2}\right]$

s) $\displaystyle\int_0^{1} \frac{x}{\sqrt{4-x^2}} \,\mathrm{d}x$ $\quad \left[2 - \sqrt{3}\right]$

t) $\displaystyle\int_0^{\pi/2} \cos^2 x \sin(2x) \,\mathrm{d}x$ $\quad \left[\frac{1}{2}\right]$

u) $\displaystyle\int_0^{\pi/2} 16\sin^4 x \,\mathrm{d}x$ $\quad [3\pi]$

v) $\displaystyle\int_0^{1} \frac{16}{8-4x^2} \,\mathrm{d}x$ $\quad \left[\sqrt{2}\ln(3+2\sqrt{2})\right]$

w) $\displaystyle\int_0^{\pi/2} 4\sin x \cos^3 x \,\mathrm{d}x$ $\quad [1]$

x) $\displaystyle\int_0^{\pi} 3\sin^3 x \,\mathrm{d}x$ $\quad [4]$

y) $\displaystyle\int_{-1}^{1} \frac{2}{\sqrt{16-4x^2}} \,\mathrm{d}x$ $\quad \left[\frac{\pi}{3}\right]$

z) $\displaystyle\int_0^{1/2} \frac{2(1+x^2)}{1-x^2} \,\mathrm{d}x$ $\quad [-1 + 2\ln 3]$

Příklad 9.3. Vypočítejte určité integrály:

a) $\displaystyle\int_0^{2} \frac{1}{(5+4x)^3} \,\mathrm{d}x$ $\quad \left[\frac{18}{4225}\right]$

b) $\displaystyle\int_0^{3} 3\sqrt{x+1} \,\mathrm{d}x$ $\quad [14]$

c) $\displaystyle\int_{-1}^{1} \frac{2}{x^2-4} \,\mathrm{d}x$ $\quad [-\ln 3]$

d) $\displaystyle\int_1^{2} e^x \left(1 + \frac{e^{-x}}{x}\right) \mathrm{d}x$ $\quad [e^2 + \ln 2 - e]$

e) $\displaystyle\int_1^{5} \frac{2\ln x}{x} \,\mathrm{d}x$ $\quad [\ln^2 5]$

f) $\displaystyle\int_0^{1} \frac{x-1}{x+1} \,\mathrm{d}x$ $\quad [1 - 2\ln 2]$

g) $\displaystyle\int_0^{\pi/8} (1 + \operatorname{tg}(2x)) \,\mathrm{d}x$ $\quad \left[\frac{\pi}{8} + \frac{1}{4}\ln 2\right]$

h) $\displaystyle\int_2^{3} \frac{6x}{x^2-1} \,\mathrm{d}x$ $\quad [9\ln 2 - 3\ln 3]$

i) $\displaystyle\int_4^{9} \frac{3(x-1)}{\sqrt{x}+1} \,\mathrm{d}x$ $\quad [23]$

j) $\displaystyle\int_0^{\pi/2} \frac{1}{1+\cos x} \,\mathrm{d}x$ $\quad [1]$

k) $\displaystyle\int_0^{2} \frac{2x}{1+x^4} \,\mathrm{d}x$ $\quad [\operatorname{arctg} 4]$

l) $\displaystyle\int_1^{2} \left(\frac{2}{x} + x^3\right) \mathrm{d}x$ $\quad [3\ln 2 - \ln 5]$

m) $\displaystyle\int_0^{1} 2\sqrt{2x+x^2} \,\mathrm{d}x$ $\quad \left[2\sqrt{3} - \ln(2+\sqrt{3})\right]$

n) $\displaystyle\int_0^{\pi/2} \frac{10}{2\cos x + 3} \,\mathrm{d}x$ $\quad \left[4\sqrt{5}\operatorname{arctg}\frac{\sqrt{5}}{5}\right]$

o) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{5+\sin x} \,\mathrm{d}x$ $\quad [\ln 6 - \ln 5]$

p) $\displaystyle\int_0^{\pi} \frac{10}{4+\cos^2 x} \,\mathrm{d}x$ $\quad \left[\sqrt{5}\pi\right]$

Příklad 9.4. Určete obsah rovinné plochy ohraničené křivkami:

a) $y = 0,\ x = -1,\ y = x^2,$ $\left[\dfrac{1}{3}\right]$

b) $y = e^x,\ x = 1,\ y = x^{-x},$ $\left[e + \dfrac{1}{e}\right]$

c) $y = 4 - x^2,\ y = 0,$ $\left[\dfrac{32}{3}\right]$

d) $yx = 1,\ x = 1,\ x = 3,\ y = 0,$ $[\ln 3]$

e) $y^2 = 2x + 1\, x - y - 1 = 0,$ $\left[\dfrac{16}{3}\right]$

f) $y(1 + x^2) = 1,\ y = \dfrac{x^2}{2},$ $\left[\dfrac{\pi}{2} - \dfrac{1}{3}\right]$

g) $y = \ln,\ x = 5,\ x = 7,\ y = 0,$ $\left[\ln \dfrac{7^7}{5^5} - 2\right]$

h) $y = |\log x|,\ x = \dfrac{1}{10},\ x = 10,\ y = 0,$ $\left[\dfrac{9.9 \ln 10 - 8.1}{\ln 10}\right]$

i) $y = -x^2 + 4x - 2,\ x + y = 2,$ $\left[\dfrac{9}{2}\right]$

j) $y = \arcsin x,\ x = 0,\ x = 1,$ $\left[\dfrac{\pi}{2} - 1\right]$

k) $y = x \sin x,\ y = 0,\ x \in \langle 0; \pi \rangle,$ $[\pi]$

l) $x = \dfrac{4}{y},\ y = 1,\ y = 4,\ x = 0,$ $[6]$

m) $y = 1 - x,\ y^2 + x^2 = 1,\ 0 \leq x,\ y > 0,$ $\left[\dfrac{\pi - 2}{4}\right]$

n) $y^2 = x,\ y = x^2$ $\left[\dfrac{1}{3}\right]$

o) $y = x^2 - x - 6,\ y = -x^2 + 5x + 14,$ $\left[\dfrac{343}{3}\right]$

p) $yx = 4,\ x + y = 5,$ $\left[\dfrac{15}{2} - 8 \ln 2\right]$

q) $y = e^{-x} \sin x,\ y = 0,\ x \in \langle 0; \pi \rangle,$ $\left[\dfrac{1 + e^{-\pi}}{2}\right]$

r) $y = \ln^2 x,\ y = \ln x$ $[3 - e]$

s) $y = \dfrac{2}{1 + x^2},\ y = x^2,$ $\left[\pi - \dfrac{2}{3}\right]$

t) $y = x^3 + x^2 - 6x,\ y = 0,\ x \in \langle -3; 3 \rangle,$ $[18]$

u) $4x^2 + 9y^2 = 36,$ $[6\pi]$

v) $y = 6x - x^2,\ y = 0$ $[36]$

w) $x^2 + y^2 = 16,\ y^2 = 6x,\ x \geq 0,$ $\left[\dfrac{4}{3}(\sqrt{3} + 4\pi)\right]$

x) $y = x^2 + 4x,\ y = x + 4,$ $\left[\dfrac{125}{6}\right]$

y) $y^2 = 2x + 1,\ x - y - 1 = 0,$ $\left[\dfrac{16}{3}\right]$

z) $y^2 = x^3,\ y = 8$ $\left[\dfrac{96}{5}\right]$

Příklad 9.5. Určete délku oblouku rovinné křivky:

a) $y^3 = x^2,\ x \in \langle 0; 1\rangle$, $\left[\dfrac{8}{27}\left(\dfrac{13\sqrt{13}}{8} - 1\right)\right]$

b) $y = \sqrt{x - x^2} - \arcsin\sqrt{x},\ x \in \langle 0; 1\rangle$, $[2]$

c) $x = a(t - \sin t),\ y = a(1 - \cos t),\ a > 0,\ t \in \langle 0; \pi\rangle$ (cykloida), $[8a]$

d) $x = a\cos^3 t,\ y = \sin^3 t,\ a > 0,\ t \in \langle 0; \pi\rangle$ (asteroida), $[6a]$

e) $y^2 = (x+1)^3,\ x \le 4$ (semikubická parabola), $\left[\dfrac{670}{27}\right]$

f) $y = \ln\dfrac{e^x + 1}{e^x - 1},\ x \in \langle 1; 2\rangle$, $\left[\ln\dfrac{e^2+1}{e}\right]$

g) $y = \ln(\sin x),\ x \in \left\langle \dfrac{\pi}{3}; \dfrac{\pi}{2}\right\rangle$, $\left[\dfrac{1}{2}\ln 3\right]$

h) $x = 2a(1 + \cos t)\cos t,\ y = 2a(1 + \cos t)\sin t,\ a > 0,\ t \in \langle 0; 2\pi\rangle$ (kardioida), $[16a]$

Příklad 9.6. Určete délku oblouku prostorové křivky:

a) $x = a\cos t,\ y = a\sin t,\ z = bt,\ a, b > 0,\ t \in \langle 0; 2\pi\rangle$ (šroubovice), $\left[2\pi\sqrt{a^2+b^2}\right]$

b) $x = t,\ y = \dfrac{1}{3}\sqrt{8t^3},\ z = \dfrac{1}{2}t^2,\ t \in \langle 0; 1\rangle$, $\left[\dfrac{3}{2}\right]$

c) $x = t - \sin t,\ y = 1 - \cos t,\ z = 4\sin\dfrac{t}{2},\ t \in \langle 0; \pi\rangle$, $[2\pi]$

d) $x = e^t,\ y = e^{-t},\ z = t\sqrt{2},\ t \in \langle 0; 1\rangle$, $\left[e - e^{-1}\right]$

Příklad 9.7. Vypočítejte hmotnost a souřadnice těžiště rovinné křivky s délkovou hustotou ρ:

a) $x = a\cos^3 t,\ y = \sin^3 t,\ y \ge 0,\ a > 0,\ \rho(t) = 1$, $\left[3a; \left[0; \dfrac{2a}{5}\right]\right]$

b) $x^2 + y^2 = r^2\ y \ge 0,\ r > 0,\ \rho(x) = 1$, $\left[\pi r; \left[0; \dfrac{2r}{\pi}\right]\right]$

c) $y = \dfrac{1}{2}(e^x + e^{-x}),\ x \in \langle -1; 1\rangle,\ \rho(x) = 1$, $\left[e + \dfrac{1}{e}; \left[0; \dfrac{e^4 + 4e^2 - 1}{4e(e^2-1)}\right]\right]$

d) $x = a(t - \sin t),\ y = a(1 - \cos t),\ a > 0,\ t \in \langle 0; 2\pi\rangle,\ \rho(t) = 1$, $\left[8a; \left[\pi a; \dfrac{4a}{3}\right]\right]$

e) $x = a\cos^3 t,\ y = \sin^3 t,\ x, y \ge 0,\ a > 0,\ \rho(t) = a\cos^3 t$, $\left[\dfrac{3a^2}{5}; \left[\dfrac{5a}{8}; \dfrac{15\pi a}{256}\right]\right]$

Příklad 9.8. Vypočítejte hmotnost a souřadnice těžiště prostorové křivky s délkovou hustotou ρ:

a) $x = a\cos t,\ y = a\sin t,\ z = bt,\ a, b > 0,\ t \in \langle 0; 2\pi\rangle,\ \rho(t) = 1$, $\left[2\pi\sqrt{a^2+b^2}; [0; 0; \pi b]\right]$

b) $x = a\cos t,\ y = a\sin t,\ z = bt,\ a, b > 0,\ t \in \langle 0; 2\pi\rangle,\ \rho(x) = 2\pi - t$, $\left[2\pi\sqrt{a^2+b^2}; \left[0; 0; \dfrac{2\pi b}{3}\right]\right]$

Příklad 9.9. Vypočítejte hmotnost a souřadnice těžiště rovinné homogenní plochy omezené křivkami:

a) $y = \dfrac{x^2}{8},\ y = 0,\ x = 8$, $\left[\dfrac{64}{3}; \left[6; \dfrac{12}{5}\right]\right]$

b) $y^2 = 4x,\ x^2 = 4y$, $\left[\dfrac{16}{3}; \left[\dfrac{9}{5}; \dfrac{9}{5}\right]\right]$

c) $y = 4 - x^2,\ y = 0$, $\left[\dfrac{32}{3}; \left[0; \dfrac{8}{5}\right]\right]$

d) $y^2 = x,\ y = x^3$, $\left[\dfrac{5}{12}; \left[\dfrac{12}{25}; \dfrac{3}{7}\right]\right]$

Příklad 9.10. Vypočítejte hmotnost a souřadnice těžiště nehomogenní rovinné plochy Ω s hustotou ρ:

a) $\Omega:\ 0 \leq y \leq \sin x,\ 0 \leq x \leq \pi,\ \rho(x) = |\cos x|,$ $\left[1; \left[\frac{\pi}{2}; \frac{1}{3}\right]\right]$

b) $\Omega:\ x^2 + y^2 \leq 4,\ 0 \leq x \leq y,\ \rho(x) = x,$ $\left[\frac{8-4\sqrt{2}}{3}; \left[\frac{3(\pi-2)}{8(2-\sqrt{2})}; \frac{3}{4(2-\sqrt{2})}\right]\right]$

c) $\Omega:\ x^2 + y^2 = ay,\ a \geq 0,\ \rho(y) = y,$ $\left[\frac{\pi a^3}{8}; \left[0; \frac{5a}{8}\right]\right]$

d) $\Omega:\ -1 \leq x \leq |y-1|,\ 0 \leq y \leq 2,\ \rho(y) = y^2,$ $\left[\frac{25}{6}; \left[-\frac{24}{125}; \frac{39}{25}\right]\right]$

Kapitola 10

Nevlastní integrál

Příklad 10.1. Vypočítejte nevlastní integrály:

a) $\displaystyle\int_1^{+\infty} \frac{2}{x^3}\,\mathrm{d}x$ $[1]$

b) $\displaystyle\int_1^{+\infty} \frac{3}{\sqrt{x^5}}\,\mathrm{d}x$ $[2]$

c) $\displaystyle\int_0^{+\infty} \frac{2x}{x^2+1}\,\mathrm{d}x$ $[+\infty]$

d) $\displaystyle\int_1^{+\infty} \frac{1}{x\sqrt{1+x^2}}\,\mathrm{d}x$ $\left[\ln(1+\sqrt{2})\right]$

e) $\displaystyle\int_0^{+\infty} \frac{4}{1+x^4}\,\mathrm{d}x$ $\left[\pi\sqrt{2}\right]$

f) $\displaystyle\int_0^{+\infty} \frac{8}{8+2x^2}\,\mathrm{d}x$ $[\pi]$

g) $\displaystyle\int_1^{+\infty} \frac{2}{x^2+2x+2}\,\mathrm{d}x$ $[\pi-\operatorname{arctg} 2]$

h) $\displaystyle\int_0^{+\infty} 3e^{-\frac{3}{10}x}\,\mathrm{d}x$ $[10]$

i) $\displaystyle\int_0^{+\infty} 4e^{-2x}\sin(2x)\,\mathrm{d}x$ $[1]$

j) $\displaystyle\int_1^{+\infty} \frac{3}{x+1}\,\mathrm{d}x$ $[+\infty]$

k) $\displaystyle\int_0^{+\infty} xe^{-x^2/2}\,\mathrm{d}x$ $[1]$

l) $\displaystyle\int_1^{+\infty} x\sin x\,\mathrm{d}x$ [diverguje]

m) $\displaystyle\int_{-\infty}^{1} \frac{1}{(x-1)(x^2+1)}\,\mathrm{d}x$ $\left[\frac{2\ln 2-\pi}{8}\right]$

n) $\displaystyle\int_2^{+\infty} \frac{\ln x}{x}\,\mathrm{d}x$ $[+\infty]$

o) $\displaystyle\int_1^{+\infty} \frac{1}{x^2(x+1)}\,\mathrm{d}x$ $[-\ln 2+1]$

p) $\displaystyle\int_{-\infty}^{0} \frac{1}{(x^2+1)(x^2+4)}\,\mathrm{d}x$ $\left[\frac{\pi}{12}\right]$

q) $\displaystyle\int_1^{+\infty} \frac{\ln x}{x^2}\,\mathrm{d}x$ $[1]$

r) $\displaystyle\int_0^{+\infty} \frac{2x}{2x^2+3}\,\mathrm{d}x$ [diverguje]

s) $\displaystyle\int_{-\infty}^{0} \frac{x}{(x^2+1)(x^2+3)}\,\mathrm{d}x$ $\left[-\frac{1}{4}\ln 3\right]$

t) $\displaystyle\int_1^{+\infty} \frac{\ln^2}{x^2}\,\mathrm{d}x$ $[2]$

u) $\displaystyle\int_2^{+\infty} \frac{1}{x\ln^2 x}\,\mathrm{d}x$ $\left[-\frac{1}{\ln 2}\right]$

v) $\displaystyle\int_0^{+\infty} \frac{2x}{(1+x)^3}\,\mathrm{d}x$ $[1]$

w) $\displaystyle\int_1^{+\infty} \frac{4\operatorname{arctg} x}{x^2}\,\mathrm{d}x$ $[\pi+2\ln 2]$

x) $\displaystyle\int_0^{+\infty} 2x^3e^{-x^2}\,\mathrm{d}x$ $[1]$

y) $\displaystyle\int_0^{+\infty} \frac{9}{1+x^3}\,\mathrm{d}x$ $\left[2\pi\sqrt{3}\right]$

z) $\displaystyle\int_0^{2} \frac{2}{\sqrt{4-x^2}}\,\mathrm{d}x$ $[\pi]$

Příklad 10.2. Vypočítejte nevlastní integrály:

a) $\displaystyle\int_1^{+\infty} \frac{\cos(\ln x)}{x}\,\mathrm{d}x$ [diverguje]

b) $\displaystyle\int_0^{\pi/2} \frac{1}{\cos^2 x}\,\mathrm{d}x$ $[+\infty]$

c) $\displaystyle\int_0^{2} (2x-1)\ln^2 x\,\mathrm{d}x$ $\left[2\ln^2 2 - 2\right]$

d) $\displaystyle\int_0^{2} \frac{1}{2-x}\,\mathrm{d}x$ $[+\infty]$

e) $\displaystyle\int_0^{2} \frac{3x^3}{\sqrt{4-x^2}}\,\mathrm{d}x$ $[16]$

f) $\displaystyle\int_0^{1} \ln x\,\mathrm{d}x$ $[-1]$

g) $\displaystyle\int_0^{1} \frac{1}{\sqrt{x}}\,\mathrm{d}x$ $[2]$

h) $\displaystyle\int_0^{1} \frac{1}{x^2-4x+3}\,\mathrm{d}x$ $[+\infty]$

i) $\displaystyle\int_1^{2} \frac{3x}{\sqrt{x-1}}\,\mathrm{d}x$ $[8]$

j) $\displaystyle\int_1^{2} \frac{1}{x\ln x}\,\mathrm{d}x$ $[+\infty]$

k) $\displaystyle\int_1^{+\infty} \frac{2\,\mathrm{arctg}\,x}{x^3}\,\mathrm{d}x$ $[1]$

l) $\displaystyle\int_1^{+\infty} \frac{4}{x^2(1+x^2)}\,\mathrm{d}x$ $[4-\pi]$

m) $\displaystyle\int_1^{+\infty} \frac{\cos(\ln x)}{x^2}\,\mathrm{d}x$ $\left[\frac{1}{2}\right]$

n) $\displaystyle\int_1^{+\infty} \frac{1}{x\sqrt{x-1}}\,\mathrm{d}x$ $[\pi]$

o) $\displaystyle\int_0^{+\infty} \frac{1}{\cos^2 x}\,\mathrm{d}x$ [není def.]

p) $\displaystyle\int_{-\infty}^{+\infty} \frac{2x}{x^2+1}\,\mathrm{d}x$ [diverguje]

q) $\displaystyle\int_{\sqrt{2}}^{+\infty} \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x$ $\left[\frac{\pi}{4}\right]$

r) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^2+2x+2}\,\mathrm{d}x$ $[\pi]$

s) $\displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{arctg}^2 x}{1+x^2}\,\mathrm{d}x$ $\left[\frac{\pi^3}{12}\right]$

t) $\displaystyle\int_{-1}^{1} \frac{e^{\frac{1}{x}}}{x^2}\,\mathrm{d}x$ $[+\infty]$

u) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x(x+1)}\,\mathrm{d}x$ [diverguje]

v) $\displaystyle\int_{-\infty}^{+\infty} e^{-|x|}\,\mathrm{d}x$ $[2]$

w) $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin x}{x^2}\,\mathrm{d}x$ [diverguje]

x) $\displaystyle\int_{-2}^{2} \frac{4x^3}{x^4-1}\,\mathrm{d}x$ [diverguje]

y) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt[3]{x}(x+1)}\,\mathrm{d}x$ $\left[\frac{2\pi}{\sqrt{3}}\right]$

z) $\displaystyle\int_0^{+\infty} \frac{1}{x^2-1}\,\mathrm{d}x$ [diverguje]